Part IIA - Labour and Unemployment Additional Notes

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Supervision 4: Integration Refresher

The unemployment component of the course uses several laws of integration, often taking these as prior knowledge. These are important techniques to understand, and know how to apply. Note that they will also reappear in the Part IIB course.

Integration by Parts

Integration by parts is useful to find the integral of a product of functions in terms of the integral of their derivative and anti-derivative. If $u = u(x)$ and $du = u'(x)dx$, while $v = v(x)$ and $dv = v'(x)dx$, then integration by parts states that:

$$\int_a^b u(x)v'(x)dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)dx$$
$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx.$$

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Leibniz's Rule

Leibniz's rule helps us deal with differentiation under the integral sign. Suppose we have an integral of the form

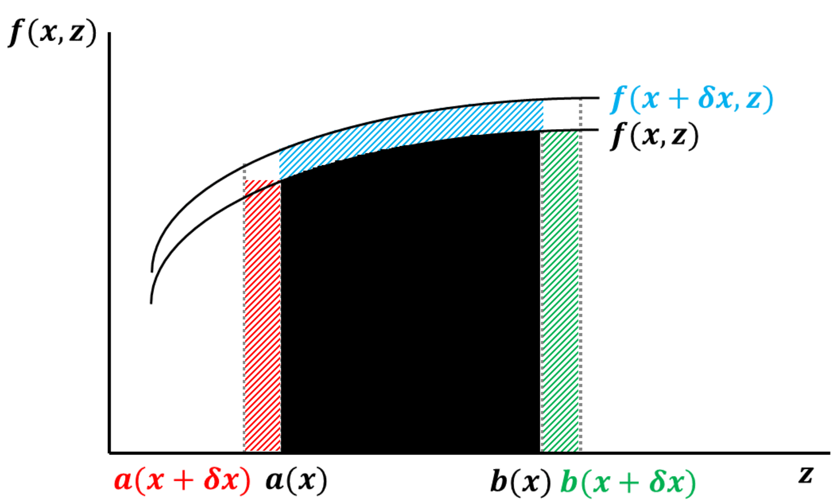
$$\int_{a(x)}^{b(x)} f(x,z)dz,$$

where $-\infty < a(x)$ and $b(x) < \infty$. It should be clear that this integral depends upon x, while any dependence upon z has been "integrated away". We may therefore be interested in the differential of this expression (with respect to x). In this case, Leibniz's rule states that the derivative of this integral is expressible as:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,z)dz\right) = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,z)dz.$$

The first two terms are necessary as, whenever x changes by an infinitesimal amount (dx) the limits of integration will respond, hence altering the integral. The partial derivative in the third expression indicates that inside the integral, only the variation of $f(x,z)$ with respect to x is considered when taking the derivative. These mathematical constructs correspond to the shaded areas in Figure 1, a highly stylised example.

Figure 1: Leibniz's Rule



Notice that if $a(x) = a$ and $b(x) = b$, where a and b are constants rather than functions of x, then we have a special case of Leibniz's rule as:

$$\frac{d}{dx}b(x) = \frac{d}{dx}a(x) = 0$$

and hence:

$$\frac{d}{dx}\left(\int_a^b f(x,z)dz\right) = \int_a^b \frac{\partial}{\partial x}f(x,z)dz.$$

As the limits of integrations are unchanged in this case, the only remaining term is the partial differential under the integral sign.

Application 1: Problem Set 4, Question 2

In the McCall model we typically define an equilibrium condition pinning down the reservation wage as:

$$h(w_r) \equiv \frac{\beta}{1-\beta}\int_{w_r}^{\infty}(w - w_r)dF(w).$$

Notice that this "integrates away" any dependence upon w, but will still depend upon w_r. In many settings we may therefore be interested in the differential

$$\frac{d}{dw_r}h(w_r) = \frac{\beta}{1-\beta}\frac{d}{dw_r}\int_{w_r}^{\infty}(w - w_r)dF(w).$$

Using Leibniz's rule this is expressible as:[1]

$$\begin{aligned}\frac{dh(w_r)}{dw_r} &= \frac{\beta}{1-\beta}\left(\int_{w_r}^{\infty} -1dF(w)\right) \\ &= -\frac{\beta}{1-\beta}(F(\infty) - F(w)) = -\frac{\beta}{1-\beta}(1 - F(w)).\end{aligned}$$

This is a particularly useful result as, for example, we may use it to show that

[1] I will sidestep the issue of the limits being finite as it turns out not to matter in this case. You can see however that this additional constraint avoids expressions such as $\infty \times 0$ which would have resulted for the first limit term.

the **slope** of the curve will alter when the subjective discount factor changes, as:

$$\begin{aligned}\frac{d}{d\beta}\frac{dh(w_r)}{dw_r} &= -\frac{d}{d\beta}\frac{\beta}{1-\beta}(1-F(w)),\\ &= -\frac{1-F(w)}{(1-\beta)^2} < 0.\end{aligned}$$

Application 2: Expected Value

Suppose an economic variable is given as:

$$y_t = \alpha + (\beta + \varepsilon_t)x_t.$$

where x_t is an input, ε_t is a (mean zero) independent random variable and α and β are known constants. Suppose we are faced with a loss function, given by:

$$L_t = \mathbb{E}_t[y_t^2|\alpha, \beta, x_t].$$

The solution to this problem is therefore through the optimal policy given by:

$$x_t = \arg\min \mathbb{E}_t[y_t^2|\alpha, \beta, x_t],$$

There are therefore two ways to proceed:

1. Integrate first, then differentiate[2]:

$$
\begin{aligned}
\frac{dL_t}{dx_t} &= \frac{d}{dx_t}\mathbb{E}_t[y_t^2|\alpha,\beta,x_t],\\
&= \frac{d}{dx_t}\mathbb{E}_t[\{\alpha+(\beta+\varepsilon_t)x_t\}^2|\alpha,\beta,x_t],\\
&= \frac{d}{dx_t}\mathbb{E}_t[\alpha^2+2\alpha(\beta+\varepsilon_t)x_t+(\beta+\varepsilon_t)^2x_t^2|\alpha,\beta,x_t],\\
&= \frac{d}{dx_t}(\alpha^2+2\alpha\beta x_t+(\beta^2+\sigma_\varepsilon^2)x_t^2),\\
&= 2\alpha\beta+2(\beta^2+\sigma_\varepsilon^2)x_t, \quad \Rightarrow x_t^* = -\frac{\alpha\beta}{\beta^2+\sigma_\varepsilon^2},
\end{aligned}
$$

   where σ_ε^2 is the variance of ε_t.

2. Use the Leibniz rule to perform differentiation first, then integration:

$$
\begin{aligned}
\frac{dL_t}{dx_t} &= \frac{d}{dx_t}\mathbb{E}_t[y_t^2|\alpha,\beta,x_t],\\
&= \mathbb{E}_t\Big[\frac{d}{dx_t}y_t^2|\alpha,\beta,x_t\Big],\\
&= \mathbb{E}_t\Big[\frac{d}{dx_t}\{\alpha+(\beta+\varepsilon_t)x_t\}^2|\alpha,\beta,x_t\Big],\\
&= \mathbb{E}_t\Big[2(\beta+\varepsilon_t)\{\alpha+(\beta+\varepsilon_t)x_t\}|\alpha,\beta,x_t\Big],\\
&= 2\alpha\beta+2(\beta^2+\sigma_\varepsilon^2)x_t, \quad \Rightarrow x_t^* = -\frac{\alpha\beta}{\beta^2+\sigma_\varepsilon^2}.
\end{aligned}
$$

[2] It should be clear than when taking the expectation of the continuous random variable ε_t, we integrate as $\mathbb{E}_t[a] \equiv \int_{-\infty}^{\infty} a d\varepsilon_t$, by definition of the expectation operator. The limits of integration are not changing as the support of the stochastic vector with which we integrate (ε_t) is independent of our input function a – we have the simplified form of the Leibniz rule. We also have infinite limits, but it turns out that is not an issue in the context.