Part IIA - Intertemporal Macroeconomics
Additional Notes

Daniel Wales

University of Cambridge

Supervision 1: CRRA Utility & Hall's Random Walk

CRRA Utility

Risk Aversion

One of the most widespread utility functions in macroeconomics is the Constant Relative Risk Aversion) utility function (CRRA):

$$U(C) = \frac{C^{1-\sigma} - 1}{1 - \sigma}.$$

The parameter, σ represents the Arrow-Pratt measure of relative risk aversion. To see this, note that:

$$\frac{\partial U}{\partial C} = C^{-\sigma},$$
$$\frac{\partial^2 U}{\partial C^2} = -\sigma C^{-\sigma-1}.$$

Hence, using the definition of the Arrow-Pratt measure of relative risk aversion and our particular utility function we have:

$$R \equiv -\frac{CU''(C)}{U'(C)} = -\frac{-\sigma C^{-\sigma-1}}{C^{-\sigma}} = \sigma.$$

Intertemporal Substitution

Under a CRRA utility function the coefficient of relative risk aversion and the Elasticity of Intertemporal Substitution, ε_{EIS}, are tightly linked. Indeed they are inversely related. This is perhaps unsurprising as σ is the only parameter describing preferences. This may be shown as follows, starting with the Euler condition from household optimisation as in lectures:

$$\begin{aligned}
U'(C_t) &= \beta(1+r)U'(C_{t+1}), && \text{(Euler condition)} \\
C_t^{-\sigma} &= \beta(1+r)C_{t+1}^{-\sigma}, && \text{(Use CRRA Utility)} \\
\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} &= \beta(1+r), && \text{(Rearrange)} \\
\ln(C_{t+1}/C_t) &= \frac{1}{\sigma}\ln\beta + \frac{1}{\sigma}\ln(1+r), && \text{(Take logs)} \\
\ln(C_{t+1}/C_t) &= \frac{1}{\sigma}\ln\beta + \frac{1}{\sigma}r. && \text{(Approximation)}
\end{aligned}$$

where the approximation in the final line, $\ln(1+r) \approx r$, is valid whenever r is small. Finally, applying the definition off the Elasticity of Intertemporal Substitution gives:

$$\varepsilon_{EIS} \equiv \frac{\partial \ln(C_{t+1}/C_t)}{\partial r} = \frac{1}{\sigma}.$$

Intratemporal Substitution

Whenever a CRRA utility function is used with multiple (separable) goods the parameter σ is also inversely related to the elasticity of substitution between these goods. For example, assuming a period utility function given by:

$$U(C,\ell) = \frac{C^{1-\sigma}-1}{1-\sigma} + \frac{\ell^{1-\sigma}-1}{1-\sigma}.$$

The elasticity of substitution between leisure and consumption may be defined as:

$$\varepsilon_{C,\ell} \equiv \frac{\partial \ln(C/\ell)}{\partial \ln(MRS_{\ell,C})}.$$

Note that under the CRRA utility function specified above:

$$MRS_{\ell,C} \equiv \frac{\frac{\partial U(C,\ell)}{\partial \ell}}{\frac{\partial U(C,\ell)}{\partial C}} = \left(\frac{\ell}{C}\right)^{-\sigma} \quad \Rightarrow \quad \ln MRS_{\ell,C} = \sigma \ln(C/\ell)$$

Hence we have that:

$$\varepsilon_{C,\ell} \equiv \frac{\partial \ln(C/\ell)}{\partial \sigma \ln(C/\ell)} = \frac{1}{\sigma}.$$

In words, the elasticity of substitution between consumption and leisure is equal to the elasticity of intertemporal substitution and both are inversely related to the coefficient of relative risk aversion.

Limiting Case (More Advanced)

Finally, the limiting case of the CRRA utility function as $\sigma \to 1$ can be understood by applying L'Hôpital's rule as:

$$\lim_{\sigma \to 1} \frac{C^{1-\sigma} - 1}{1-\sigma} + \frac{\ell^{1-\sigma} - 1}{1-\sigma} = \ln(C) + \ln(\ell).$$

i.e., an additively separable log utility function. To see this first note that we have a problem since the limit as $\sigma \to 1$ may be written as:

$$\frac{0}{0} + \frac{0}{0},$$

which is undefined. For this reason we employ L'Hôpital's rule which states that the limit of a ratio may be rewritten as the limit of the derivatives of that ratio, provided these exist. Specifically:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided both $f'(x)$ and $g'(x)$ exist in the limit. In our application we may therefore set:

$$f(\sigma) = e^{(1-\sigma)\ln C} - 1 + e^{(1-\sigma)\ln \ell} - 1 \qquad \text{and} \qquad g(\sigma) = 1 - \sigma,$$
$$f'(\sigma) = -\ln C \cdot e^{(1-\sigma)\ln C} - \ln \ell \cdot e^{(1-\sigma)\ln \ell} \qquad \text{and} \qquad g'(\sigma) = -1.$$

Together with L'Hôpital's rule this implies:

$$\begin{aligned}\lim_{\sigma\to 1}\frac{C^{1-\sigma}-1}{1-\sigma}+\frac{\ell^{1-\sigma}-1}{1-\sigma} &= \lim_{\sigma\to 1}\frac{-\ln C\cdot e^{(1-\sigma)\ln C}-\ln\ell\cdot e^{(1-\sigma)\ln\ell}}{-1},\\ &= \lim_{\sigma\to 1}\frac{-\ln C-\ln\ell}{-1},\\ &= \ln C+\ln\ell.\end{aligned}$$

As both $f'(x)$ and $g'(x)$ exist in the limit this is our solution.[1]

Hall's Random Walk

To appreciate the contribution of Hall (1978) we must first understand the statistical concept of a random walk. A martingale sequence is a series of random variables, X_1, X_2, X_3, $\cdots$, which satisfy the following two conditions, at any time period, t:

$$\mathbb{E}_t[X_t] < \infty,$$

$$\mathbb{E}_t[X_{t+1}] = X_t.$$

A random walk requires one additional assumption. Finite variance of an iid error term, such that:

$$X_{t+1} = X_t + \varepsilon,$$

where, for example, $\varepsilon \overset{iid}{\sim} N(0,\sigma^2)$.

Doppelhofer (2009) outlines one of the contributions of Hall (1978), focussing on the easier case when utility is assumed to be quadratic. However, the paper makes two contributions with the second functional form assumed to be CRRA, as discussed above.

[1]Note that, from time to time, you may see utility written down as:

$$\frac{C^{1-\sigma}-1}{1-\sigma}, \quad \text{or} \quad \frac{C^{1-\sigma}}{1-\sigma} \quad \text{or} \quad C^{1-\sigma}.$$

All are obviously related and the fact that utility is **ordinal** means these functions will result in the same economic behaviour. The function we use above is the preferred one, as the parameter σ is then directly linked to risk aversion, and the limit as $\sigma \to 1$ is clearly defined as the logarithm.

Basic Setting (Quadratic Utility)

Hall (1978) (Corollary 3) begins with the Euler equation of the household problem, makes an assumption about the functional form of utility and derives an interesting theoretical result:

$$\begin{aligned}
U'(C_t) &= \beta(1+r)\mathbb{E}_t[U'(C_{t+1})], && \text{(Euler condition)}\\
a - C_t &= \beta(1+r)\mathbb{E}_t[a - C_{t+1}], && \text{(Assumption 1: } U(C) = -\tfrac{1}{2}(a-C)^2\text{)}\\
\mathbb{E}_t[C_{t+1}] &= \frac{\beta(1+r)-1}{\beta(1+r)}a + \frac{1}{\beta(1+r)}C_t, && \text{(Rearrange)}\\
C_{t+1} &= \beta_0 + \beta_1 C_t + \varepsilon_t. && \text{(Regression)}
\end{aligned}$$

where $a > 0$ in the assumed utility function is the bliss level of consumption. The final rearrangement highlight how this assumption results in a random walk with drift and should display $\beta_1 \equiv \frac{1}{\beta(1+r)} > 0$, while the drift term $\beta_0 \equiv \frac{\beta(1+r)-1}{\beta(1+r)}$ has no clear sign.

A second assumption is often added to simplify the process, namely:

$$\beta(1+r) = 1.$$

In this case the Euler condition reduces further to become:

$$\begin{aligned}
\mathbb{E}_t[C_{t+1}] &= C_t,\\
C_{t+1} &= C_t + \varepsilon_t. && \text{(Regression)}
\end{aligned}$$

which now depicts a martingale sequence for household consumption (as the variance property is not assumed). Imposing stationary variance properties, to configure a random walk then allows Hall to empirically test errors in the above equation.

Advanced Setting (CRRA Utility)

In a second setting (Corollary 4) Hall (1978) performs a similar analysis using a CRRA utility function. In that case the Euler condition becomes:

$$
\begin{aligned}
U'(C_t) &= \beta(1+r)\mathbb{E}_t[U'(C_{t+1})], && \text{(Euler condition)}\\
C_t^{-\sigma} &= \beta(1+r)\mathbb{E}_t[C_{t+1}^{-\sigma}], && \text{(Assumption 1: } U(C)=\tfrac{C^{1-\sigma}-1}{1-\sigma}\text{)}\\
\mathbb{E}_t[C_{t+1}^{-\sigma}] &= \frac{C_t^{-\sigma}}{\beta(1+r)}, && \text{(Rearrange)}\\
C_{t+1}^{-\sigma} &= \beta_1 C_t^{-\sigma} + \varepsilon_t. && \text{(Regression)}
\end{aligned}
$$

where we should expect $\beta_1 \equiv \frac{1}{\beta(1+r)} > 0$. He then tests the relationship assuming different values of σ (Table 1) to produce a close fit.

An alternative approach could have been to use Jensen's inequality to show:

$$
\begin{aligned}
U'(C_t) &= \beta(1+r)\mathbb{E}_t[U'(C_{t+1})], && \text{(Euler condition)}\\
C_t^{-\sigma} &= \beta(1+r)\mathbb{E}_t[C_{t+1}^{-\sigma}], && \text{(Assumption 1: } U(C)=\tfrac{C^{1-\sigma}-1}{1-\sigma}\text{)}\\
-\ln[\beta(1+r)] &= \ln \mathbb{E}_t\Big[\Big(\frac{C_{t+1}}{C_t}\Big)^{-\sigma}\Big], && \text{(Rearrange and logs)}\\
-\ln[\beta(1+r)] &\geq \mathbb{E}_t\Big[\ln\Big(\frac{C_{t+1}}{C_t}\Big)^{-\sigma}\Big], && \text{(Jensen's Inequality)}\\
\mathbb{E}_t[\ln C_{t+1}] &\geq \frac{1}{\sigma}\ln[\beta(1+r)] + \ln C_t, && \text{(Rearrange (assuming } \sigma>0\text{))}\\
\ln C_{t+1} &\geq -\frac{1}{\sigma}\ln[\beta_1] + \ln C_t + \varepsilon_t. && \text{(Regression)}
\end{aligned}
$$

where when taking the logarithm inside the expectation operator Jensen's inequality applies since the logarithm is a concave function and hence $\mathbb{E}_t[\ln X_t] \leq \ln \mathbb{E}_t[X_t]$. The equality arises whenever X_t is almost surely constant. Alternatively equality is approximate whenever the growth rate of consumption is small, since this arises as the first component in a Taylor expansion around that point. This again results in a random walk with drift, though this time for the logarithm of consumption.

Prudence

One problem with the traditional analysis used in Hall (1978), particularly the more well known basic setting, is the potential absence of risk aversion and a precautionary saving motive (prudence) in shaping household preferences. Assuming a form of utility which disregards this important behaviour of households is problematic. This is particularly evident during times of economic crisis, such as the global financial crisis or Covid-19 pandemic.

Taking the standard utility function with Constant Relative Risk Aversion with, $\sigma > 0$ will be enough to ensure households are risk averse, and prudence is switched on, i.e. using the Advanced Setting in Hall (1978).

Table 1: Utility Functions and their Derivatives

	CRRA	Quadratic
$U(C)$	$\frac{C^{-\sigma}-1}{1-\sigma}$	$-\frac{1}{2}(a-C)^2$
$\partial U/\partial C$	$C^{-\sigma}$	$a-C$
$\partial^2 U/\partial C^2$	$-\sigma C^{-\sigma-1}$	a
$\partial^3 U/\partial C^3$	$\sigma(\sigma+1)C^{-\sigma-2}$	0

- An individual will dislike risk (displaying risk aversion) whenever both $\partial U/\partial C > 0$ and $\partial^2 U/\partial C^2 < 0$.
  - In the CRRA case this arises whenever $\sigma > 0$
  - This never arises for the quadratic utility form above as $a > 0$ is assumed.
- An individual will save more if faced with additional risk (displaying **prudence**) whenever $\partial^3 U/\partial C^3 > 0$. This additional saving resulting from prudence is known as **precautionary** savings.
  - In the CRRA case this arises whenever $\sigma(\sigma+1) > 0$.
  - This never arises for quadratic utility as $\partial^3 U/\partial C^3 = 0$.

References

Doppelhofer, G. (2009). Intertemporal Macroeconomics.

Hall, R. E. (1978). Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence. *Journal of Political Economy 86*(6), 971–987.