Part IIB - Continuous Time Models
Additional Notes

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Supervision 1: Continuous Time Models

This note provides greater detail for a number of distinctions between continuous and discrete time models. In lectures we have been introduced to continuous time models, but may wonder in which cases these are preferred to their more traditional discrete time counterparts. Discrete time models have the following advantages over their continuous time counterparts:

- All real economic and financial data are ultimately only available in discrete form, so discrete models can be calibrated easily to real data points.
- Computers work in discrete units, so a discrete model can be simulated by a computer.
- Continuous time is a special case of discrete time (in the limit as the length of each period approaches zero) so discrete models are more general.

However this comes at the cost of the following:

- Continuous time models are sometimes easier to work with and solve analytically (using pen-and-paper). Closed form solutions may exist even when they do not for discrete time counterparts.
- Relatedly, continuous time models may therefore be quicker to solve (computationally).

- Equations used in continuous time models sometimes have clearer economic interpretation (e.g. Euler condition or growth rate approximations, see application below).
- Continuous time models may eradicate some determinacy properties of equilibria.
- Habits or convention in a subfield may mean one form is strictly preferred to another (for the purpose of comparability etc). For example in Part IIB, growth theory is discussed in continuous time while RBC theory is analysed in discrete time.

Ultimately modern economists need to be comfortable using both forms.

Growth Rates

One important area where the distinction between continuous and discrete time variables arises is in the computation of growth rates. In this section we will prove two key results.

1. The growth rate of a variable is approximately its difference in logarithms.
2. The time derivative of the logarithm for a variable is equal to its growth rate.

Notation

In discrete time, the change in a variable, X_t, between two periods is often abbreviated to:

$$\Delta X_t \equiv X_t - X_{t-1},$$

while its growth rate may be calculated as:

$$\gamma_{X_t} \equiv \frac{\Delta X_t}{X_t} = \frac{X_t - X_{t-1}}{X_t}.$$

The equivalent expressions in continuous time are the variable's **rate** of change, which is often abbreviated using "dot" notation to:

$$\dot{X}(t) \equiv \frac{dX(t)}{dt},$$

and the (proportional) rate of growth given as:

$$g_X(t) \equiv \frac{\dot{X}(t)}{X(t)} = \frac{dX(t)/dt}{X(t)}.$$

In each case growth rates may be multiplied by 100 to be expressed in percentage terms. Including t in both γ_{X_t} and $g_X(t)$ allows for the possibility that growth rates may not be constant.

We may prove that the continuous time growth rate represents the limit of the growth rate in discrete time when the time period, initially 1 unit, is changed to become Δt and then shortened.

$$\lim_{\Delta t \to 0} \gamma_{X_t} \equiv \lim_{\Delta t \to 0} \left[\frac{X_t - X_{t-\Delta t}}{\Delta t} \frac{1}{X_t} \right] = \frac{dX(t)}{dt} \frac{1}{X(t)} = g_X(t).$$

Growth Rate as Approximately the Log Difference

When economists analyse growth rates they tend to use the difference in logarithms, the logic behind that argument is as follows.

In discrete time, rearranging the above expression for the growth rate of X_t we observe:

$$X_t = (1 + \gamma_{X_t}) X_{t-1}.$$

Taking logs of this equation we may write:

$$\ln X_t = \ln(1 + \gamma_{X_t}) + \ln X_{t-1}$$

such that the difference in logarithms:

$$\hat{X}_t \equiv \ln X_t - \ln X_{t-1} = \ln(1 + \gamma_{X_t}) \approx \gamma_{X_t},$$

where the approximation is reasonable for small γ_{X_t}.

Time derivative of the Logarithm of a Variable

Alternatively, the continuous time limit of the above relationship is exactly the time derivative of the logarithm of the variable X_t as:

$$\begin{aligned}\lim_{\Delta t\to 0}\hat{X}_t &\equiv \lim_{\Delta t\to 0}\left[\frac{\ln X_t - \ln X_{t-\Delta t}}{\Delta t}\right],\\ &= \frac{d\ln X(t)}{dt},\\ &= \frac{d\ln X(t)}{dX(t)}\frac{dX(t)}{dt},\\ &= \frac{1}{X(t)}\frac{dX(t)}{dt},\\ &= \frac{\dot{X}(t)}{X(t)},\\ &\equiv g_X(t).\end{aligned}$$

such that the time derivative of the logarithm of $X(t)$ (i.e. $\frac{d\ln X(t)}{dt}$) is equal to the growth rate, $g_X(t)$.

We may illustrate this through an application assuming that output is produced according to a Cobb Douglas production function.

Discrete Time	Continuous Time	
$Y_t = K_t^{\alpha}L_t^{1-\alpha}$	$Y(t) = K(t)^{\alpha}L(t)^{1-\alpha},$	(Production Function)
$y_t = \alpha k_t + (1-\alpha)\ell_t$	$y(t) = \alpha k(t) + (1-\alpha)\ell(t),$	(Take logs)
$\hat{y}_t = \alpha\hat{k}_t + (1-\alpha)\hat{\ell}_t$	$\frac{d\ln Y(t)}{dt} = \alpha\frac{d\ln K(t)}{dt} + (1-\alpha)\frac{d\ln L(t)}{dt},$	(First Difference / Differente)
$\gamma_{Y_t} \approx \alpha\gamma_{K_t} + (1-\alpha)\gamma_{L_t},$	$g_Y(t) = \alpha g_K(t) + (1-\alpha)g_L(t),$	(Final Result)

where lower case variables represent logarithms and the final result in discrete time is an approximation that applies for small growth rates but is exact in continuous time.