Speed of Convergence in Solow Model

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Consider the Solow growth model expressed in terms of capital per effective worker $\tilde{k}$, with Cobb-Douglas production function $\tilde{y} = \tilde{k}^{\alpha}$, savings rate s, depreciation rate δ, rate of population growth n, and rate of technological progress g. Then the fundamental equation of motion is

$$\dot{\tilde{k}} = s\tilde{k}^{\alpha} - (\delta + g + n)\,\tilde{k} \tag{1}$$

In the steady state, $\dot{\tilde{k}} = 0$, which implies

$$\tilde{k}^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}} \tag{2}$$

Using (1), the growth rate of $\tilde{k}$ can be expressed as follows:

$$g_{\tilde{k}} \equiv \frac{\dot{\tilde{k}}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (\delta + g + n) \equiv G\left(\tilde{k}\right) \tag{3}$$

Note that in the steady state, $g_{\tilde{k}}^* = G\left(\tilde{k}^*\right) = 0$.

To compute the speed of convergence, take the first-order Taylor approximation of $G\left(\tilde{k}\right)$ around the steady state k^*:

$$G\left(\tilde{k}\right) \approx G\left(\tilde{k}^*\right) + G'\left(\tilde{k}^*\right)\left(\tilde{k} - \tilde{k}^*\right)$$

Substituting the derivative $G'\left(\tilde{k}\right)$ of (3), it follows that

$$G\left(\tilde{k}\right) \approx (\alpha - 1)\, s\left(\tilde{k}^*\right)^{\alpha-1}\left(\frac{\tilde{k} - \tilde{k}^*}{\tilde{k}^*}\right)$$

Finally, substituting (2) and simplifying gives

$$g_{\tilde{k}} \approx -(1-\alpha)(\delta + g + n)\left(\frac{\tilde{k} - \tilde{k}^*}{\tilde{k}^*}\right) \tag{4}$$

The term $\beta \equiv (1-\alpha)(\delta + g + n)$ is the speed of convergence. It measures how quickly $\tilde{k}$ increases when $\tilde{k} < \tilde{k}^*$. The growth rate of $\tilde{k}$ depends on the speed of convergence β and the percentage difference between $\tilde{k}$ and $\tilde{k}^*$.

Taking the first-order Taylor approximation of $\ln\tilde{k}$ around the steady state $\tilde{k}^*$ yields

$$\ln\tilde{k} \approx \ln\tilde{k}^* + \frac{1}{\tilde{k}^*}\left(\tilde{k} - \tilde{k}^*\right) \quad \Leftrightarrow \quad \frac{\tilde{k} - \tilde{k}^*}{\tilde{k}^*} \approx \left(\ln\tilde{k} - \ln\tilde{k}^*\right)$$

Substituting this into (4) gives

$$g_{\tilde{k}} \approx -(1-\alpha)(\delta + g + n)\left(\ln\tilde{k} - \ln\tilde{k}^*\right) \tag{5}$$

The production function implies $\ln\tilde{y} = \alpha\ln\tilde{k}$, so using the fact that $\frac{d\ln x}{dt} = \frac{d\ln x}{dx}\frac{dx}{dt} = \frac{\dot{x}}{x}$,

$$g_{\tilde{y}} \equiv \frac{\dot{\tilde{y}}}{\tilde{y}} = \frac{d\ln\tilde{y}}{dt} = \alpha\frac{d\ln\tilde{k}}{dt} = \alpha\frac{\dot{\tilde{k}}}{\tilde{k}} = \alpha g_{\tilde{k}}$$

As a result, it follows from (5) that

$$g_{\tilde{y}} \approx -(1-\alpha)(\delta + g + n)(\ln\tilde{y} - \ln\tilde{y}^*)$$

Again, $\beta \equiv (1-\alpha)(\delta + g + n)$ is the speed of convergence. It measures how quickly $\tilde{y}$ increases when $\tilde{y} < \tilde{y}^*$. The growth rate of $\tilde{y}$ depends on the speed of convergence β and the log-difference between $\tilde{y}$ and $\tilde{y}^*$.