EC421: International Economics

International Macroeconomics

Additional Notes: Trade Elasticity, Complementarity and Substitutability

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Lecture 3: Complementarity and Substitutability

Assume we have a CRRA utility function and CES Consumption Aggregator:

1. CRRA utility function:
$$U_t = \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$
2. CES Consumption Aggregator:
$$C_t = \left[\alpha^{1/\omega} C_{H,t}^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_{F,t}^{\frac{\omega-1}{\omega}}\right]^{\frac{\omega}{\omega-1}}.$$

Note that C_H and C_F are substitutes if the marginal utility of one good is decreasing in the quantity of the other.

Step 1 Write out the functional form of utility in terms of both goods, C_H and C_F:

$$U = \frac{\left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{\omega(1-\sigma)}{\omega-1}} - 1}{1-\sigma}$$

Step 2 Take the partial derivative of the felicity function, with respect to C_H:

$$\begin{aligned}
\frac{\partial U}{\partial C_H} &= \frac{\frac{\omega(1-\sigma)}{\omega-1}}{1-\sigma} \cdot \left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{\omega(1-\sigma)}{\omega-1}-1} \cdot \alpha_H^{1/\omega} \cdot \frac{\omega-1}{\omega} \cdot C_H^{\frac{\omega-1}{\omega}-1}, \\
&= \left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{\omega(1-\sigma)}{\omega-1}-1} \cdot \alpha_H^{1/\omega} \cdot C_H^{\frac{\omega-1}{\omega}-1}, \\
&= \left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{1-\omega\sigma}{\omega-1}} \cdot \alpha_H^{1/\omega} \cdot C_H^{-1/\omega},
\end{aligned}$$

where the final step rewrites the powers.

Step 3 Take the partial derivate with respect to C_F:

$$\begin{aligned}
\frac{\partial^2 U}{\partial C_H \partial C_F} &= \frac{1-\omega\sigma}{\omega-1} \cdot \left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{1-\omega\sigma}{\omega-1}-1} \cdot \alpha_H^{1/\omega} \cdot C_H^{-1/\omega} \cdot \alpha_F^{1/\omega} \cdot \frac{\omega-1}{\omega} \cdot C_F^{\frac{\omega-1}{\omega}-1}, \\
&= \frac{1-\omega\sigma}{\omega} \cdot \left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{1-\omega\sigma}{\omega-1}-1} \cdot \alpha_H^{1/\omega} \cdot C_H^{-1/\omega} \cdot \alpha_F^{1/\omega} \cdot C_F^{\frac{\omega-1}{\omega}-1}, \\
&= \frac{1-\omega\sigma}{\omega} \cdot \left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{2-\omega(\sigma+1)}{\omega-1}} \cdot \alpha_H^{1/\omega} \cdot C_H^{-1/\omega} \cdot \alpha_F^{1/\omega} \cdot C_F^{-1/\omega}, \\
&= \frac{1-\omega\sigma}{\omega} \cdot \left[\alpha^{1/\omega} C_H^{\frac{\omega-1}{\omega}} + \alpha_F^{1/\omega} C_F^{\frac{\omega-1}{\omega}}\right]^{\frac{2-\omega(\sigma+1)}{\omega-1}} \cdot \left[\frac{\alpha_H \alpha_F}{C_H C_F}\right]^{1/\omega}.
\end{aligned}$$

Step 4 Inspect sign:

Clearly the sign of this expression depends critically on the term $1 - \omega\sigma$. All other terms are positive due to parameter restrictions, and assumptions surrounding the non-negativity of consumption. This therefore give the following interpretation.

$$\begin{aligned}
&\text{When} \quad \omega\sigma < 1 \quad \rightarrow \quad \frac{\partial^2 U}{\partial C_H \partial C_F} > 0 \quad \text{and the two goods are complements,} \\
&\text{When} \quad \omega\sigma > 1 \quad \rightarrow \quad \frac{\partial^2 U}{\partial C_H \partial C_F} < 0 \quad \text{and the two goods are substitutes.}
\end{aligned}$$