EC421: International Economics

International Macroeconomics

Problem Set 4

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1 The Hodrick-Prescott Filter

Consider the HP filter as outlined during the lecture, for the set of data $\{y_t\}_{t=1}^{T}$. The loss function is summarised by:

$$L = \sum_{t=1}^{T}(y_t - y_t^T)^2 + \lambda \sum_{t=2}^{T-1}[(y_{t+1}^T - y_t^T) - (y_t^T - y_{t-1}^T)]^2,$$

where $\lambda \geq 0$ and our objective is to extract the trend $\{y_t^T\}_{t=1}^{T}$ from the original data.

(a) Provide an intuitive explanation for this loss function, and how for how it aims to separate data from trend.

Answer: The HP filter weighs together two statistical components of a data series. It applies a weight of 1 to the cyclical component of the data, i.e. how far trend and data move apart. It then places a weight of λ on the smoothness of the statistical trend, i.e. how much the trend itself moves over time. By minimising the sum of these losses, the HP filter extracts trend from raw data.

(b) Find an analytical solution to this loss function for the case when $\lambda = 0$, and provide an intuitive explanation for this result.

Answer: When $\lambda = 0$, the loss function may be rewritten as:

$$L_{\lambda=0} = \sum_{t=1}^{T}(y_t - y_t^T)^2.$$

Clearly the solution to this loss function will be for $y_t^T = y_t \; \forall t$. At this point $L_{\lambda=0} = 0$, which is the minimum value this loss function may take.

In other words, the trend and the actual series would be identical, as the HP filter places no weight on the volatility of the trend, and it is therefore unconstrained from matching the actual data. In this case there is no cyclical component.

(c) Find an analytical solution to this loss function for the case when $\lambda =\rightarrow \infty$, and provide an intuitive explanation for this result.

Answer: When $\lambda \rightarrow \infty$, the loss function may be rewritten as:

$$L_{\lambda\rightarrow\infty} = \sum_{t=2}^{T-1} [(y_{t+1}^T - y_t^T) - (y_t^T - y_{t-1}^T)]^2,$$

The solution to this functions is any linear time trend, with $y_t^T = at + b$. Using this in the objective function we observe:

$$L_{\lambda\rightarrow\infty} = \sum_{t=2}^{T-1} [(a(t+1) + b - at - b) - (at + b - a(t-1) - b)]^2 = \sum_{t=2}^{T-1} [a - a]^2 = 0,$$

which again must minimise the loss function. Intuitively, here we have that all of the weight in the loss function is placed variations in the trend component. Thus the "best" trend component is simply a straight line, of any form. In particular, distance of the trend from the data does not matter.

(d) Download quarterly, seasonally-adjusted, US real GDP data for 1947Q1 - 2018Q1.[1] Using a computer programme of your choice (MatLab, Stata, EViews...) firstly, take the logarithm of the raw data, and then calculate the value of HP-filtered real GDP. Plot both series, and the accompanying cyclical data (as a percentage of trend).

Answer: Shown in Figures 1 and 2.

[1] This may be found either from the St. Louis Federal Reserve data services at https://fred.stlouisfed.org/ or the Bureau of Economic Analysis at https://www.bea.gov/.

Figure 1: Log US GDP

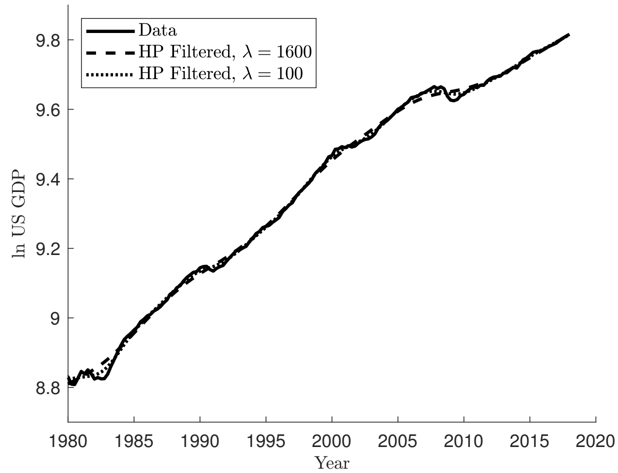
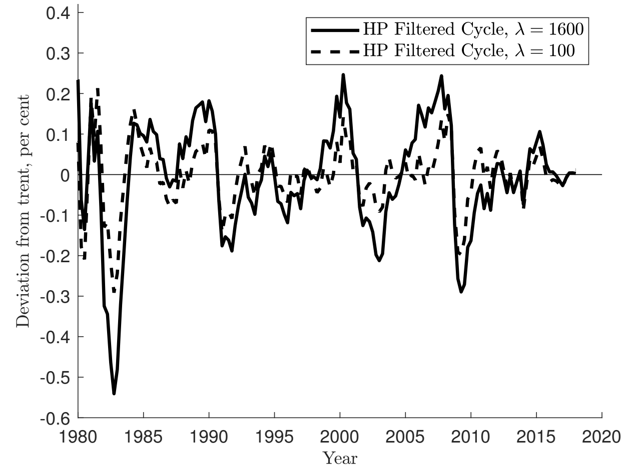


Figure 2: Cyclical Components of HP Filter



Sources: Federal Reserve Board, BEA, Bank of England and ONS.

(e) Repeat this exercise (using the same quarterly data) for the HP filter but now setting $\lambda = 100$. Comment on the cyclical properties of the data.

Answer: Again, answers shown in Figures 1 and 2. We notice that, as argued in the previous parts of the question, when we lower the value of λ the trend moves closer to the raw data. This has the effect of lowering the variance of cyclical components. If a greater (relative) loss is now derived from cyclical variance, the HP filter minimises this. Hence cycles are less severe under a HP filter with $\lambda = 100$.

2 International Real Business Cycles in a Small Open Economy

This question follows the structure in Mendoza (1991) and Chapter 4 of Schmitt-Grohé and Uribe (2017). Consider the planner problem for a small open economy. Preferences for a representative household are defined over consumption, c_t, and labour supply, ℓ_t given by:

$$U_t(c_t, \ell_t) = \mathbb{E}_t\Big[\sum_{s=t}^{\infty} \beta^{s-t} \frac{(c_s - \frac{\ell_s^{\omega}}{\omega})^{1-\gamma} - 1}{1-\gamma}\Big],$$

where $\gamma > 0$ and $\omega > 0$. The representative household maximises utility subject to a period-t budget constraint, give as:

$$b_t + c_t + i_t + \Phi(k_t - k_{t-1}) = y_t + (1 + r_{t-1})b_{t-1},$$

where b_t denotes the representative agent's net holdings of international financial assets and the function $\Phi(x)$ captures capital stock, k_t, adjustment costs with a functional form given as $\Phi(x) = \frac{\phi}{2}(x)^2$. r represents the real rate of interest on net foreign assets, and is explained further, below. Output, y_t, is produced by means of a linearly homogeneous productions functions that takes the capital and labour as inputs such that:

$$y_t = z_t k_{t-1}^{\alpha} \ell_t^{1-\alpha}.$$

where z_t is an exogenous technology process. The law of motion for domestic capital is given as:

$$k_t = (1-\delta)k_{t-1} + i_t,$$

and, finally, the exogenous technology process is assumed to follow an AR(1) structure in logs with:

$$\ln(z_t) = \rho \ln(z_{t-1}) + \varepsilon_t,$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$.

(a) Set up the problem facing the social planner in this model. What are the choice variables? What are the endogenous and exogenous state variables? Do you have enough of each? Eliminate the constraints to reduce the problem as much as possible.

Answer: The social planner chooses output, consumption, investment, labour supply, the capital stock and level of net foreign assets. Namely:

$$\{y_t, c_t, i_t, \ell_t, k_t, b_t\},$$

to maximise the utility function, subject to the budget constraint, production and capital accumulation equation. The technology process is exogenous.

By eliminating the production function and capital accumulation equation we can reduce the problem to the following Lagrangian:

$$\max_{c_s,\ell_s,k_s,b_s} \mathbb{E}_t\Big[\sum_{s=t}^{\infty} \beta^{s-t} \frac{(c_s - \frac{\ell_s^{\omega}}{\omega})^{1-\gamma} - 1}{1-\gamma} + \lambda_s \Big(z_s k_{s-1}^{\alpha} \ell_s^{1-\alpha} + (1+r_{s-1})b_{s-1} - b_s - c_s - k_s + (1-\delta)k_{s-1} - \frac{\phi}{2}(k_s - k_{s-1})^2\Big)\Big],$$

such that the social planner will have four first order conditions to determine four variables:

$$\{c_t, \ell_t, k_t, b_t\},$$

while output and investment may then be found using the production function and capital accumulation condition.

(b) Firstly, calculate the optimality conditions for the planner's problem. Then, use these equilibrium relationships to eliminate the Lagrange multiplier, λ_t from the problem.

Answer: There will be four first order conditions of the problem, which may be written as:

$$u_c^{'} \equiv \left(c_t - \frac{\ell_t^\omega}{\omega}\right)^{-\sigma} = \lambda_t, \qquad \text{(FOC, } c_t\text{)}$$

$$-u_\ell^{'} \equiv \left(c_t - \frac{\ell_t^\omega}{\omega}\right)^{-\sigma} \ell_t^{\omega-1} = (1-\alpha)\lambda_t z_t k_{t-1}^{\alpha} \ell_t^{-\alpha}, \qquad \text{(FOC, } \ell_t\text{)}$$

$$\lambda_t[1+\phi(k_t - k_{t-1})] = \beta \mathbb{E}_t[\lambda_{t+1}(\alpha z_{t+1} k_t^{\alpha-1} \ell_{t+1}^{1-\alpha} + 1 - \delta - \phi(k_{t+1} - k_t))], \qquad \text{(FOC, } k_{t+1}\text{)}$$

$$\lambda_t = \beta(1+r_t)\mathbb{E}_t[\lambda_{t+1}]. \qquad \text{(FOC, } b_t\text{)}$$

Alongside these first order conditions, the budget constraint (i.e. the first order condition with respect to λ_t) is also an equilibrium equation such that the problem is fully specified as 5 unknown variables in 5 equations. When we eliminate λ_t from the first order conditions we have:

$$\left(c_t - \frac{\ell_t^\omega}{\omega}\right)^{-\sigma} = \beta(1+r_t)\mathbb{E}_t\left[\left(c_{t+1} - \frac{\ell_{t+1}^\omega}{\omega}\right)^{-\sigma}\right],$$

$$\ell_t^{\omega-1} = (1-\alpha) z_t k_{t-1}^{\alpha} \ell_t^{-\alpha},$$

$$\left(c_t - \frac{\ell_t^\omega}{\omega}\right)^{-\sigma}\Big[1+\phi(k_t - k_{t-1})\Big] = \beta \mathbb{E}_t\left[\left(c_{t+1} - \frac{\ell_{t+1}^\omega}{\omega}\right)^{-\sigma}\Big(\alpha z_{t+1} k_t^{\alpha-1} \ell_{t+1}^{1-\alpha} + 1 - \delta - \phi(k_{t+1} - k_t)\Big)\right],$$

$$b_t + c_t + k_t - (1-\delta)k_{t-1} + \frac{\phi}{2}(k_t - k_{t-1})^2 = z_t k_{t-1}^{\alpha} \ell_t^{1-\alpha} + (1+r_{t-1})b_{t-1},$$

where the budget constraint has been included as the final equation.

(c) We will solve the problem in two steps. Firstly, we consider the steady state (stationary point) of the system. We consider a non-stochastic stationary point, such that $z_t = \bar{z}$. Re-write the optimality conditions from part (c) which will determine the steady state.

Answer: The non-stochastic steady state is the solution to the system of non-linear equations given as:

$$1 = \beta(1+r),$$

$$\ell^{\omega-1} = (1-\alpha)\bar{z} k^{\alpha} \ell^{-\alpha},$$

$$1 = \beta\Big[\alpha \bar{z} k^{\alpha-1} \ell^{1-\alpha} + 1 - \delta\Big],$$

$$c + \delta k = \bar{z} k^{\alpha} \ell^{1-\alpha} + rb.$$

(d) A well-documented problem in this setting is that the Euler equation will not help in determining any of the endogenous variables, and therefore the system has an infinite number of solutions. To side-step this issue, we will assume an endogenous real interest rate for the small open economy which depends critically on the level of net foreign assets chosen by households. However this dependence of the real interest rate on the level of net foreign assets is not internalised by households, who simply observe that:

$$r_t = r^* + \psi_1(\exp^{\bar{b}-b_t} - 1),$$

where the real interest rate now includes a risk premium term which is a convex function with respect to external debt. The real interest rate for the small open economy therefore becomes an endogenous variable, our number of unknown variables increases by 1 as does our number of equilibrium conditions. Show how this relationship, combined with the Euler equation can may help determine an endogenous variable in the steady state.

Answer: We simply combine the Euler equation and endogenous real interest rate to show that, together, they determine the equilibrium level of net foreign assets. This is true as:

$$1 = \beta(1 + r),$$

becomes:

$$1 = \beta(1 + r^* + \psi_1(\exp^{\bar{b}-b_t} - 1)),$$

which, in turn, then implies:

$$b_t = \bar{b} - \ln\Big(\frac{1}{\psi_1}\Big[\frac{1}{\beta} - (1 + r^*)\Big] + 1\Big),$$

such that net foreign assets are clearly determined as a function of entirely exogenous variables.

(e) We now solve for the steady state explicitly. Use Matlab, along with its in-built fsolve function, to write a compute programme to solve for the steady state. Use the parameters as given in Table 1. *Hint: Rewrite the equilibrium conditions into a vector of equations which may then be set equal to 0.*

Answer: See the file steady_solver.m.

(f) **Extension:** Using Dynare, write a computer program to solve the model.[2] Again, Table 1 provides the parameter values of the model.

[Hint: In the Dynare folder 'examples', you will find a number of models which you should be able to run easily on your computers. As a starting point, you can manipulate one of these to accommodate the differences with the model in this exercise. Notice the file 'bkk.mod' will solve the model in Backus, Kehoe and Kydland (1992), but note that this may not be the most simplistic model to manipulate.]

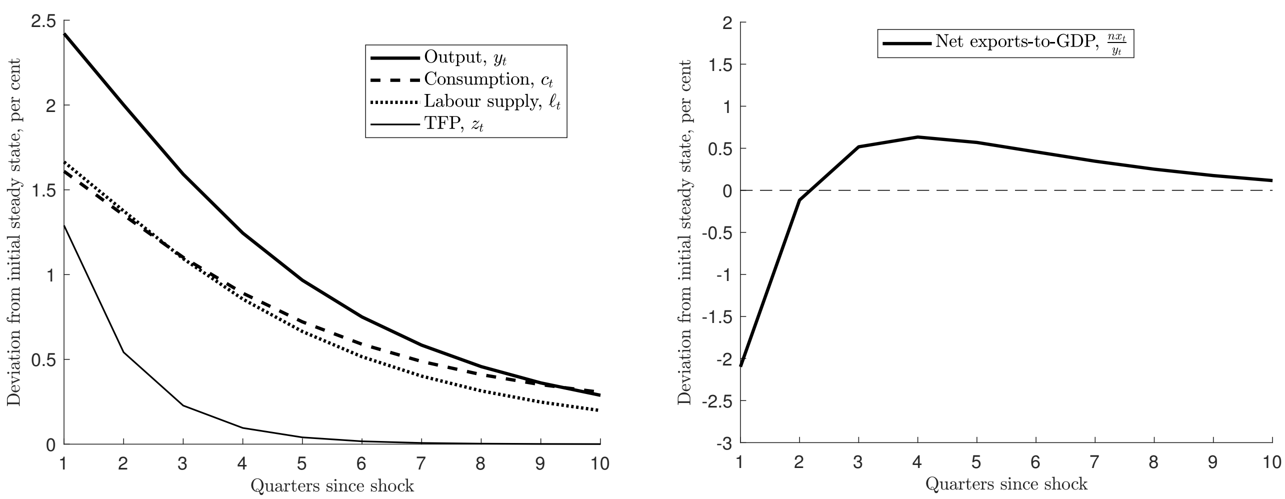
Answer: See the file 'mendoza1991.mod'. Impulse responses are given in Figure 3 show that this model generates procyclical consumption and labour supply, while net exports are countercyclical - matching the international business cycle stylised facts.

[2] Dynare is downloadable for free at http://www.dynare.org/. It works with Matlab and Octave on Windows, Mac, and Linux platforms. See the Dynare webpage for more details, including the manual and user guide that will explain in details its functionalities.

Table 1: Parameter values

β	=	0.96	Discount factor
γ	=	2	Coefficient of relative risk aversion
δ	=	0.1	Depreciation rate
α	=	0.32	Capital income share
r^*	=	0.04	World real interest rate
$\bar{b}$	=	0.7442	NFA without interest rate premium
ψ_1	=	$\bar{b}/100$	Debt sensitivity of interest rate
ϕ	=	0.028	Capital adjustment cost parameter
ω	=	1.455	IES of labour supply, $(\frac{1}{\omega-1})$
ρ	=	0.42	AR(1) technology parameter
σ_ε^2	=	0.0129^2	Technology shock variance

Figure 3: Impulse Response Functions



Source: Dynare replication of SGU.

A Matlab Code

A.1 Main.m Routine

```
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % % %           Mendoza (1991)          % % % % % % % % % %
% % % % % % % % % % %          D. Wales (2018)          % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

%%  % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % % %             Pre-amble             % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
clear;          clc;                                    % Tidy up.
cd('C:\Users\ddgw2\Documents\Teaching\LSE International Economics\Daniel Wales - 2018\Models\Mendoza1991
addpath 'S:\dynare\4.4.3\matlab';        % Link Matlab to Dynare.
options = optimset('MaxIter',1e7,'MaxFunEvals',1e7,'Algorithm','trust-region-dogleg','Display','on');
ftsz    = 11; % Set chart font sizes.

%%  % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % % %      Solution to a Single Case      % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % % % % % % % % %
% % %   Static  % % %
% % % % % % % % % % %
XX      = 0.5*ones(9,1);                         % Guess endogenous variables.
steady0 = steady_solver(XX);                     % Check steady_solver works!
steady0 = fsolve('steady_solver',XX,options);    % Solve for steady state.


% Prepare for Dynare (Log-linear solution).
steady0                 = log(steady0);
save steady0 steady0

% % % % % % % % % % %
% % %  Dynamics % % %
% % % % % % % % % % %
%dynare mendoza1991_Linear  noclearall;
dynare mendoza1991  noclearall;
```

A.2 Steady Solver.m Sub-Routine

```
function [F] = steady_solver(XX)
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 
% % % % % % % % % %         Steady State Solver         % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 

%   1) Assumes gross interest rates.
%   2) Solves for the social planner's solution.
%   3) NFA are defined as debt, rather than assets.

%%  % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 
% % % % % % % % % %     Unpack Endogenous Variables     % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 
c        = XX(1);
ell      = XX(2);
k        = XX(3);
b        = -XX(4);
z        = XX(5);
r        = XX(6);

% Definitial variables
y        = XX(7);
ii       = XX(8);
nx       = XX(9);


%%  % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 
% % % % % % % % % %          Unpack Parameters          % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 
% SGU Table 4.1
cbeta    = 0.96;
cgamma   = 2;
cdelta   = 0.1;
calpha   = 0.32;
crstar   = 0.04;
cbbar    = 0.7442;
cpsi1    = cbbar/1000;
cphi     = 0.028;
comega   = 1.455;
crho     = 0.42;
csigEE   = 0.0129;

%%  % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % 
```

```
% % % % % % % % % %     Steady State Equation Errors    % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% Euler condition:
F(1) = 1 - cbeta*r;

% Consumption-Leisure Trade-off:
F(2) = (ell^(comega-1)) - (1-calpha)*z*(k^(calpha))*(ell^(-calpha));

% Capital Euler condition:
F(3) = 1 - cbeta*(calpha*z*(k^(calpha-1))*(ell^(1-calpha)) + 1 - cdelta);

% Budget Constraint:
F(4) = c + cdelta*k - z*(k^(calpha))*(ell^(1-calpha)) - (r-1)*b;

% Technology:
F(5) = z - 1;

% Endogenous interest rate:
F(6) = (r-1) - crstar - cpsi1*(exp(cbbar-b)-1);

% Definitions:
% Output:
F(7) = y - z*(k^(calpha))*(ell^(1-calpha));
% Investment:
F(8) = ii - cdelta*k;
% Net Exports to output ratio:
F(9) = nx - (y - c - ii)/y;

end
```

A.3 Mendoza1991.mod Dynare-Routine

```
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % % %            Mendoza (1991)           % % % % % % % % % %
% % % % % % % % % % %           D. Wales (2018)           % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % % %          Defining Variables         % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
var c ell k b z r y ii nx;

varexo EE;

parameters cbeta cgamma cdelta calpha crstar cbbar cpsi1 cphi comega crho csigEE;

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % % %             Calibration             % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% SGU Table 4.1
cbeta   = 0.96;
cgamma  = 2;
cdelta  = 0.1;
calpha  = 0.32;
crstar  = 0.04;
cbbar   = 0.7442;
cpsi1   = cbbar/1000;
cphi    = 0.028;
comega  = 1.455;
crho    = 0.42;
csigEE  = 0.0129;

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % % %                Model                % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% Social Planner's Solution.
model;
% Euler condition:
((c - ((ell^(comega))/comega))^(-cgamma)) = ...
cbeta*r(-1)*((c(+1) - ((ell(+1)^(comega))/comega))^(-cgamma));

% Consumption-Leisure Trade-off:
(ell^(comega-1)) = (1-calpha)*z*(k(-1)^(calpha))*(ell^(-calpha));
```

```
% Capital Euler condition:
((c - ((ell^(comega))/comega))^(-cgamma))*(1+cphi*(k-k(-1))) = ...
((c(+1) - ((ell(+1)^(comega))/comega))^(-cgamma))*cbeta*(calpha*z*(k^(calpha-1))*(ell^(1-calpha)) + ...
1 - cdelta - cphi*(k(+1)-k));

% Budget Constraint:
c + b + k - (1-cdelta)*k(-1) + (cphi/2)*((k - k(-1))^2) =...
 z*(k(-1)^(calpha))*(ell^(1-calpha)) + r*b(-1);

% Technology:
log(z) = crho*log(z(-1)) + EE;

% Endogenous interest rate:
(r-1) = crstar + cpsi1*(exp(cbbar-b)-1);

% Output:
y = z*(k(-1)^(calpha))*(ell^(1-calpha));
% Investment:
ii = k - (1-cdelta)*k(-1);
% Net Exports:
nx = (y - c - ii - (cphi/2)*((k - k(-1))^2))/y;
end;

% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % % % %            Computation            % % % % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %
steady;
check;

shocks;
var EE; stderr csigEE;
end;

stoch_simul(order=1,graph);
```

References

Mendoza, E. G. (1991). Real Business Cycles in a Small Open Economy. *American Economic Review 81*(4), 797–818.

Schmitt-Grohé, S. and M. Uribe (2017). *Open Economy Macroeconomics*. Princeton, NJ: Princeton University Press.