EC421: International Economics

International Macroeconomics

Problem Set 2

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1 Overlapping Generations and Production

Consider the overlapping generations model with no government discussed in Lecture 2. The economy is small with respect to the rest of the world (so the economy takes the world real interest rate r as given) and populated by overlapping generations who live for two periods (young age y and old age o). Young households supply one unit of labour inelastically in exchange for a wage w_t while old households do not work. Young households are born with no assets and can save via a competitive mutual fund, which at time t issues claims a_t that promise to pay a net interest rate r in $t+1$. The size of the cohort born in t is $N_t = (1+n)N_{t-1}$.

(a) Write down the per-period budget constraints and the intertemporal budget constraint of a representative household born in period t.

Answer: The per-period budget constraints are:

$$c_t^y + a_t = w_t N_t$$
$$c_{t+1}^o = (1+r)a_t.$$

Substituting a_t out, we can write the intertemporal budget constraint as:

$$c_t^y + \frac{c_{t+1}^o}{1+r} = w_t N_t.$$

(b) A representative firm produces output y_t with a Cobb-Douglas technology

$$y_t = k_t^\alpha (z_t \ell_t)^{1-\alpha},$$

where $\alpha \in (0, 1)$, z_t is a labour-augmenting productivity parameter that grows at rate g (i.e, $z_t = (1+g)z_{t-1}$), and k_t and ℓ_t are the amount of capital and labour, respectively, used in production. The firm rents capital from the mutual fund and hires labour from the household in competitive input markets, taking as given the factor prices r_t^k and w_t. Assume capital does not depreciate. What are the optimality conditions for capital labour? Argue that $r_t^k = r^k = r,\ \forall t$.

Answer: The firm maximizes profits subject to its technology constraint, that is, it solves:

$$\max_{k_t, \ell_t}\ y_t - w_t \ell_t - r_t^k k_t,$$

subject to:

$$y_t = k_t^\alpha (z_t \ell_t)^{1-\alpha}.$$

The first order conditions for this problem are:

$$r_t^k = \alpha k_t^{\alpha-1} (z_t \ell_t)^{1-\alpha},$$
$$w_t = (1-\alpha) k_t^\alpha z_t^{1-\alpha} \ell_t^{-\alpha}.$$

Because the mutual fund owns the capital stock and operates in perfect competition, the return on renting capital to the firm must equal its outside option, which is to lend in international financial markets at the interest rate r. Therefore, it must be the case that:

$$r_t^k = r^k = r,$$

so that the first order condition for capital becomes:

$$r = \alpha k_t^{\alpha-1} (z_t \ell_t)^{1-\alpha}.$$

(c) Let b_t denote the stock of net foreign assets at time t. Write down the asset market resource constraint.

Answer: The household desired stock of assets at time t is a_t. If the economy were closed to trade in financial assets with the rest of the world, these resources would have to correspond to the amount of physical assets in the economy. That is, if $b_t = 0 \Rightarrow a_t = k_t$. If the economy can trade in financial assets with the rest of the world, the equality between savings and investment (and hence between stock of assets demanded by the private sector and stock of assets available domestically) needs not hold. In particular, if the representative households wants to accumulate assets in excess of the available physical capital stock, she can save in internationally traded bonds. That is, the asset market resource constraint is

$$a_t = k_t + b_t.$$

(d) Write down the labour market clearing condition.

Answer: Because the labour market is competitive, the wage will adjust to the point in which $\ell_t = N_t$.

(e) Assume utility is log in consumption and the individual discount factor is $\beta \in (0, 1)$. Write down the set of equations that characterize the competitive equilibrium in efficiency units (that is, $\tilde{x}_t \equiv x_t/(z_t N_t)$ for any variable x_t in the model, except for the wage, for which $\tilde{w}_t \equiv w_t/z_t$).

Answer: A competitive equilibrium for this economy is a sequence of quantities and prices such that households and firms optimize subject to their relevant constraints, and all markets clear. Households' optimisation requires:

$$\frac{\tilde{c}^o_{t+1}}{\tilde{c}^y_t} = \frac{\beta(1+r)}{(1+g)(1+n)},$$

and:

$$\tilde{c}^y_t + \frac{(1+g)(1+n)\tilde{c}^o_{t+1}}{1+r} = \tilde{w}_t.$$

Firms' optimisation requires:

$$r = \alpha \tilde{k}_t^{\alpha-1},$$
$$\tilde{w}_t = (1-\alpha)\tilde{k}_t^{\alpha},$$

with:

$$\tilde{y}_t = \tilde{k}_t^{\alpha}.$$

Finally, the asset market equilibrium requires:

$$\tilde{a}_t = \tilde{k}_t + \tilde{b}_t,$$

where from the youngs' budget constraint:

$$\tilde{a}_t = \tilde{w}_t - \tilde{c}^y_t.$$

(f) Solve the model (i.e., find a solution for variables in efficiency units).

Answer: We can find a solution for the model by observing that the first order condition for capital gives:

$$\tilde{k}_t = \tilde{k} = \left(\frac{\alpha}{r}\right)^{\frac{1}{1-\alpha}}.$$

From the first order condition for labour, we obtain:

$$\tilde{w}_t = \tilde{w} = (1-\alpha)\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}},$$

while from technology we have:

$$\tilde{y}_t = \tilde{y} = \left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}}.$$

Combining the Euler equation and the intertemporal budget constraint of the household, we have:

$$\tilde{c}_t^y = \frac{\tilde{w}_t}{1+\beta} = \frac{1-\alpha}{1+\beta}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}} = \tilde{c}^y,$$

which then implies:

$$\tilde{c}_t^o = \tilde{c}^o = \frac{\beta(1+r)}{(1+g)(1+n)}\frac{1-\alpha}{1+\beta}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}}.$$

Finally, from the youngs' budget constraint, we have:

$$\tilde{a}_t = \tilde{a} = \frac{\beta(1-\alpha)}{1+\beta}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}},$$

and from the asset market equilibrium:

$$\tilde{b}_t = \tilde{b} = \frac{\beta(1-\alpha)}{1+\beta}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha}{r}\right)^{\frac{1}{1-\alpha}}.$$

(g) Derive an expression for the equilibrium current account (defined as $ca_t = b_t - b_{t-1}$) as a fraction of GDP y_t.

Answer: The current account as a fraction of GDP is $ca_t/y_t = \tilde{ca}_t/\tilde{y}_t$. Therefore, using the definition of the current account, we can rewrite:

$$\frac{ca_t}{y_t} = \frac{\widetilde{ca}_t}{\tilde{y}_t} = \frac{\tilde{b}_t}{\tilde{y}_t} - \frac{1}{(1+n)(1+g)}\frac{\tilde{y}_{t-1}}{\tilde{y}_t}\frac{\tilde{b}_{t-1}}{\tilde{y}_{t-1}} = \left[\frac{(1+n)(1+g)-1}{(1+n)(1+g)}\right]\frac{\tilde{b}}{\tilde{y}}.$$

From the solution in part (f) of the question, we have:

$$\frac{\tilde{b}}{\tilde{y}} = \frac{\beta(1-\alpha)}{1+\beta} - \frac{\alpha}{r}.$$

Therefore, the solution for the current account as a fraction of GDP is:

$$\frac{ca_t}{y_t} = \frac{\widetilde{ca}_t}{\tilde{y}_t} = \frac{\widetilde{ca}}{\tilde{y}} = \left[\frac{n+g+ng}{(1+n)(1+g)}\right]\left[\frac{\beta(1-\alpha)}{1+\beta} - \frac{\alpha}{r}\right].$$

(h) Assume $\beta(1+r) = 1$. What can you conclude about the current account as a fraction of GDP in this model? What are the effects of a increase in the population growth rate? And of productivity growth? *[Hint: Think about reasonable values for the real interest rate to infer a range of values for β. Reasonable values for α follow from the observation that, in most countries, the labor share is between 0.7 and 0.5.]*

Answer: If $\beta(1+r) = 1 \Rightarrow r\beta = 1-\beta$. Therefore, the second bracket in the expression for the current account is positive if and only if:

$$r\beta(1-\alpha) = (1-\beta)(1-\alpha) > \alpha(1+\beta) \Rightarrow 1-\beta > 2\alpha.$$

Recall that α represents the capital share (from the first order condition for the firm, you can show that $r^k k/y = \alpha$. For most countries, α is a number between 0.3 and 0.5. Taking the midpoint of that range (i.e. $\alpha = 0.4$) implies that the condition above is satisfied when $1-\beta > 0.8 \Rightarrow \beta < 0.2$. Because standard values for β are bigger than 0.9, the current account as a fraction of GDP will generally be negative for reasonable values of the parameters. Notice that the current account is increasing in β, that is, more patient countries will have relatively smaller deficits.

You can also show that the effects of an increase in population growth depend on the last bracket of the expression for the current account

$$\frac{\partial(ca_t/y_t)}{\partial n} = \frac{1}{(1+n)^2(1+g)}\left[\frac{\beta(1-\alpha)}{1+\beta} - \frac{\alpha}{r}\right],$$

and so do the effects of an increase in output growth:

$$\frac{\partial(ca_t/y_t)}{\partial g} = \frac{1}{(1+n)(1+g)^2}\left[\frac{\beta(1-\alpha)}{1+\beta} - \frac{\alpha}{r}\right].$$

Therefore, for reasonable calibrations, these partial derivatives are negative.

2 Perpetual Youth

Consider a small open economy inhabited by a continuum of identical agents of measure one who face uncertainty about their survival. In every period, independently of age, agents survive with probability γ and die (after consuming) with probability $1-\gamma$. Agents care about per-period consumption c_t according to the utility function $u(c_t)$, with $u' > 0$ and $u'' < 0$, and discount the future at rate $\beta \in (0, 1/\gamma)$.

(a) Show that expected utility for the representative agent of this economy from the perspective of time t (before consuming) is:

$$\mathbb{E}_t U_t = \sum_{j=0}^{\infty} (\beta\gamma)^j u(c_{t+j})$$

Answer: The probability of dying after consuming in t is $1-\gamma$. The probability of dying after consuming in $t+1$ is $\gamma(1-\gamma)$. The probability of dying after consuming in $t+2$ is $\gamma^2(1-\gamma)$, and so on. Expected utility is a weighted average of utility over the different possible lifespans, where the weights are given by the probabilities of a specific lifespan:

$$\mathbb{E}_t U_t = (1-\gamma)u(c_t) + \gamma(1-\gamma)[u(c_t) + \beta u(c_{t+1})] + \gamma^2(1-\gamma)[u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2})] + ...$$

Collecting terms, we can rewrite:

$$\mathbb{E}_t U_t = (1-\gamma)u(c_t)(1+\gamma+\gamma^2+...)+(1-\gamma)\beta\gamma u(c_{t+1})(1+\gamma+\gamma^2+...)+(1-\gamma)\beta^2\gamma^2 u(c_{t+2})(1+\gamma+\gamma^2+...)+...$$

Because the series in parenthesis converges to $(1-\gamma)^{-1}$, we obtain that expected utility is:

$$\mathbb{E}_t U_t = \sum_{j=0}^{\infty} (\beta\gamma)^j u(c_{t+j}).$$

(b) Suppose that at the start of each period t a new generation, consisting again of a continuum of members of size one, is born. Show that total population size in this economy is constant and equal to $(1-\gamma)^{-1}$.

Answer: At time t, all members of the generation born in that period are still alive. Of those born at $t-1$, γ are still alive. Of those born at $t-2$, only $\gamma - (1-\gamma)\gamma = \gamma^2$ are still alive. And so on. Therefore, total population alive at t (and hence at any point in time) is:

$$1 + \gamma + \gamma^2 + ... = \frac{1}{1-\gamma}.$$

(c) Suppose a competitive insurance industry sells contracts at time t that pay agents $(1+r)/\gamma$ if they are alive at $t+1$ and nothing if they die. Agents who want to borrow can do so at the same interest rate.

Show that if the insurance industry holds all of residents assets and finances all of their borrowing, earning or paying the world interest rate r, then it must break even.

Answer: The insurance industry pays pays $(1+r)/\gamma$ to those who survive from one period to the next. But only a fraction γ of the population does survive between each two periods. Therefore, the insurance industry cost per contract is $1+r$, which equals the return on investing in international financial markets. Hence, profits are zero.

(d) Agents are born with no initial wealth (no bequest motive). Argue that agents prefer to buy the insurance contracts in 3. than a standard one-period bond that pays a net interest rate r. Write down the flow budget constraint for an agent born in period ν assuming the agent receives an endowment y_t^ν and pays lump-sum taxes τ_t^ν at time t.

Answer: Agents prefer the insurance contract because it compensates them for the risk of dying (higher return). Denoting with b_t^ν the amount of insurance contract the agent buys at t, the flow budget constraint is:

$$c_t^\nu + b_t^\nu = \frac{1+r}{\gamma} b_{t-1}^\nu + y_t^\nu - \tau_t^\nu.$$

(e) Write the intertemporal budget constraint after imposing the relevant terminal condition.

Answer: Following the steps highlighted in Lectures 1 and 2 (after appropriately accounting for the modified return), the intertemporal budget constraint is

$$\sum_{j=0}^{\infty} \left(\frac{\gamma}{1+r}\right)^j c_{t+j}^\nu = \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{j=0}^{\infty} \left(\frac{\gamma}{1+r}\right)^j (y_{t+j}^\nu - \tau_{t+j}^\nu),$$

where the appropriate transversality condition is:

$$\lim_{T\to\infty} \left(\frac{\gamma}{1+r}\right)^T b_{t+T}^\nu = 0.$$

(f) Assume that $u(c_t^\nu) = \ln c_t^\nu$. Calculate aggregate private consumption as a function of aggregate private net foreign assets, output, and taxes. Derive the law of motion of net foreign assets. *[Hint: Aggregate variables at time t are the sum from t to $-\infty$ of the variable for each cohort, weighted by the number of agents of that cohort alive at time t.]*

Answer: The Euler equation for consumption is standard, because the effective individual discount factor is $\beta\gamma$.

$$\frac{c_{t+1}^\nu}{c_t^\nu} = \beta(1+r).$$

Replacing into the intertemporal budget constraint gives:

$$c_t^{\nu} = (1 - \beta\gamma) \left[\frac{1+r}{\gamma} b_{t-1}^{\nu} + \sum_{j=0}^{\infty} \left(\frac{\gamma}{1+r} \right)^j (y_{t+j}^{\nu} - \tau_{t+j}^{\nu}) \right].$$

To derive aggregate consumption, we can integrate over alive at time t. Note that, for the law of large numbers, only a fraction of agents γ will carry their assets from $t-1$ to t. Therefore, we can write aggregate consumption:

$$c_t = (1 - \beta\gamma) \left[(1+r) b_{t-1} + \sum_{j=0}^{\infty} \left(\frac{\gamma}{1+r} \right)^j (y_{t+j} - \tau_{t+j}) \right].$$

Similarly, from the households' budget constraint, we can obtain:

$$b_t = (1+r) b_{t-1} + y_t - c_t - \tau_t.$$

(g) Assume that $y_t = y$ and $\tau_t = \tau$, $\forall t$. Derive an expression that characterizes the dynamics of net foreign assets. Assuming $\beta\gamma(1+r) < 1$, describe graphically the dynamics of net foreign.

Answer: Plugging the solution for consumption into the last expression of the previous question and imposing constant output and taxes yields:

$$b_t = (1+r) b_{t-1} + y - \tau - (1 - \beta\gamma) \left[(1+r) b_{t-1} + \frac{1+r}{1+r-\gamma} (y - \tau) \right],$$

which after simplifications gives:

$$b_t = \beta\gamma(1+r) b_{t-1} + \gamma \left[\frac{\beta(1+r) - 1}{1+r-\gamma} \right] (y - \tau).$$

In steady state, net foreign assets are constant and given by:

$$b^{ss} = \frac{\gamma[\beta(1+r) - 1](y - \tau)}{(1+r-\gamma)[1 - \beta\gamma(1+r)]}.$$

The equation that determines the evolution of net foreign assets can be represented in a diagram with b_{t-1} on the horizontal axis and b_t on the vertical axis. Given the restriction in the question, the evolution of net foreign assets is an upward sloping line, with slope less than one. Assuming further $\beta(1+r) > 1$, this line intercepts the vertical axis at $\gamma[\beta(1+r) - 1](y - \tau)/(1+r-\gamma)$ and the 45^o degree line at the steady state value b^{ss}.

Suppose initial net foreign assets are higher than steady state. Because the evolution of net foreign assets is captured by a line that is flatter than the 45^o degree line, next period net foreign assets will decrease. Hence, net foreign assets will converge toward their steady state value. A similar reasoning suggests that we have convergence also when we start with an initial stock of net foreign assets lower than the steady state.[1]

(h) Suppose government spending is equal to zero in every period but the government starts with some debt d, which it finances through a uniform tax $\tau = rd(1-\gamma)$ on everyone alive. How do changes in d affect steady state net foreign assets and consumption?

Answer: Because the size of the population is $(1-\gamma)^{-1}$, total taxes are just rd. Substituting into steady state net foreign assets, we obtain:

$$b^{ss}(d) = \frac{\gamma[\beta(1+r)-1]}{(1+r-\gamma)[1-\beta\gamma(1+r)]}(y-rd).$$

For consumption, we have:

$$c^{ss} = (1-\beta\gamma)(1+r)\left[b + \frac{1}{1+r-\gamma}\right](y-\tau).$$

Substituting the solution steady state net foreign assets and simplifying, we obtain:

$$c^{ss}(d) = \frac{(1-\beta\gamma)(1+r)(1-\gamma)}{(1+r-\gamma)[1-\beta\gamma(1+r)]}(y-rd).$$

An increase in government debt unequivocally depresses private consumption. The effect on steady state net foreign assets depends on whether $\beta(1+r)$ is larger or smaller than one. Note that we only impose the restriction $\beta\gamma(1+r) < 1$. Therefore, the effect of higher government debt on net foreign assets is ambiguous.

[1]See OR, figure 3.9.