EC421: International Economics

International Macroeconomics

Additional Notes: Log Linearisation

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Problem Set 3: Consumption Bundles and Price Indices

Assume we have identified the following equation to be log-linearised:

1. Real exchange rage:

$$Q_t = \left[\frac{\mu^* + (1-\mu^*)\mathcal{T}_t^{1-\eta}}{\mu + (1-\mu)\mathcal{T}_t^{1-\eta}}\right]^{\frac{1}{1-\eta}} \left[\frac{\gamma + (1-\gamma)X_t^{*,1-\theta}}{\gamma + (1-\gamma)X_t^{1-\theta}}\right]^{\frac{1}{1-\theta}}.$$

Step 1 Take logs of the equation:

$$\ln Q_t = \left(\frac{1}{1-\eta}\right)\Big[\ln[\mu^* + (1-\mu^*)\mathcal{T}_t^{1-\eta}] - \ln[\mu + (1-\mu)\mathcal{T}_t^{1-\eta}]\Big] + \left(\frac{1}{1-\theta}\right)\Big[\ln[\gamma + (1-\gamma)X_t^{*,1-\theta}] - \ln[\gamma + (1-\gamma)X_t^{1-\theta}]\Big].$$

Step 2 Where needed, replace variables with $a = \exp^{\ln a}$:

$$\ln Q_t = \left(\frac{1}{1-\eta}\right)\Big[\ln[\mu^* + (1-\mu^*)\exp^{(1-\eta)\ln \mathcal{T}_t}] - \ln[\mu + (1-\mu)\exp^{(1-\eta)\ln \mathcal{T}_t}]\Big] + \left(\frac{1}{1-\theta}\right)\Big[\ln[\gamma + (1-\gamma)\exp^{(1-\theta)\ln X_t^*}] - \ln[\gamma + (1-\gamma)\exp^{(1-\theta)\ln X_t}]\Big].$$

Step 3 Take derivatives:

$$
\begin{aligned}
\frac{\partial \ln Q_t}{\partial \ln \mathcal{T}_t} &= \Big(\frac{1}{1-\eta}\Big)\left[\frac{(1-\eta)(1-\mu^*)\exp^{(1-\eta)\ln \mathcal{T}_t}}{\mu^* + (1-\mu^*)\exp^{(1-\eta)\ln \mathcal{T}_t}} - \frac{(1-\eta)(1-\mu)\exp^{(1-\eta)\ln \mathcal{T}_t}}{\mu + (1-\mu)\exp^{(1-\eta)\ln \mathcal{T}_t}}\right],\\
\frac{\partial \ln Q_t}{\partial \ln X_t} &= -\Big(\frac{1}{1-\theta}\Big)\left[\frac{(1-\theta)(1-\gamma)\exp^{(1-\theta)\ln X_t}}{\gamma + (1-\gamma)\exp^{(1-\theta)\ln X_t}}\right],\\
\frac{\partial \ln Q_t}{\partial \ln X_t^*} &= \Big(\frac{1}{1-\theta}\Big)\left[\frac{(1-\theta)(1-\gamma)\exp^{(1-\theta)\ln X_t^*}}{\gamma + (1-\gamma)\exp^{(1-\theta)\ln X_t^*}}\right].
\end{aligned}
$$

Step 3 Evaluate derivatives at steady state ($\bar{\mathcal{T}} = 1$, $\bar{X} = 1$, and $\bar{X}^* = 1$):

$$
\begin{aligned}
\frac{\partial \ln Q_t}{\partial \ln \mathcal{T}_t}|_{\mathcal{T}_t=\bar{\mathcal{T}}} &= \frac{(1-\mu^*)}{\mu^* + (1-\mu^*)} - \frac{(1-\mu)}{\mu + (1-\mu)} = \mu - \mu^*,\\
\frac{\partial \ln Q_t}{\partial \ln X_t}|_{X_t=\bar{X}} &= -\frac{(1-\gamma)}{\gamma + (1-\gamma)} = -(1-\gamma),\\
\frac{\partial \ln Q_t}{\partial \ln X_t^*}|_{X_t^*=\bar{X}^*} &= \frac{(1-\gamma)}{\gamma + (1-\gamma)} = (1-\gamma).
\end{aligned}
$$

Step 4 Note, and apply, First Order Taylor expansion formula (in multivariate form):

$$
\begin{aligned}
&f(a) \approx f(\bar{a}) + f'(\bar{a})(a - \bar{a}),\\
&\ln Q_t \approx \ln \bar{Q} + \frac{\partial \ln Q_t}{\partial \ln \mathcal{T}_t}|_{\mathcal{T}_t=\bar{\mathcal{T}}}(\ln \mathcal{T}_t - \ln \bar{\mathcal{T}}) + \frac{\partial \ln Q_t}{\partial \ln X_t}|_{X_t=\bar{X}}(\ln X_t - \ln \bar{X}) + \frac{\partial \ln Q_t}{\partial \ln X_t^*}|_{X_t^*=\bar{X}^*}(\ln X_t^* - \ln \bar{X}^*)
\end{aligned}
$$

Step 5 Substitute in the required formulae:

$$
\ln Q_t \approx \ln \bar{Q} + (\mu - \mu^*)(\ln \mathcal{T}_t - \ln \bar{\mathcal{T}}) - (1-\gamma)(\ln X_t - \ln \bar{X}) + (1-\gamma)(\ln X_t^* - \ln \bar{X}^*)
$$

Step 5 Rewrite in deviation form, $\hat{a} \equiv \ln a - \ln \bar{a}$:

$$
\begin{aligned}
\ln Q_t - \ln \bar{Q} &\approx (\mu - \mu^*)(\ln \mathcal{T}_t - \ln \bar{\mathcal{T}}) - (1-\gamma)(\ln X_t - \ln \bar{X}) + (1-\gamma)(\ln X_t^* - \ln \bar{X}^*),\\
\hat{Q}_t &\approx (\mu - \mu^*)\hat{\mathcal{T}}_t + (1-\gamma)(\hat{X}_t^* - \hat{X}_t).
\end{aligned}
$$