EC421: International Economics

International Macroeconomics

Additional Notes: Linear Aggregation Procedure

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1 Lecture 2: Perpetual Youth (Weil Model)

Assume we have identified the following four properties of the model:

1. Total population, N_t and, within this, the size of each cohort, ν.

$$N_t = \underset{\nu=0}{1} + \underset{\nu=1}{n} + n\underset{\nu=2}{(n+1)} + \cdots + n\underset{\nu=t}{(n+1)}^{t-1} = (1+n)^t.$$

2. The period-t budget constraint facing cohort ν:

$$b_t^\nu + c_t^\nu = (1+r)b_{t-1}^\nu + y_t - \tau_t.$$

3. The intertemporal budget constraint (IBC) for a given cohort, ν:

$$\sum_{s=t}^{\infty} \frac{c_s^\nu}{(1+r)^{s-t}} = (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}.$$

4. The Euler equation for a given cohort, ν:

$$c_{t+1}^\nu = \beta(1+r)c_t^\nu.$$

Step 1 Combine the Euler equation and intertemporal budget constraint to produce the consumption policy function for a given cohort, ν:

$$\begin{aligned}
\sum_{s=t}^{\infty} \frac{c_s^\nu}{(1+r)^{s-t}} &= (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}, \\
c_t^\nu + \frac{c_{t+1}^\nu}{1+r} + \frac{c_{t+2}^\nu}{(1+r)^2} + \cdots &= (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}, \\
c_t^\nu + \frac{\beta(1+r)c_t^\nu}{1+r} + \frac{\beta^2(1+r)^2 c_t^\nu}{(1+r)^2} + \cdots &= (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}, \\
c_t^\nu + \beta c_t + \beta^2 c_t^\nu + \cdots &= (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}, \\
c_t^\nu (1 + \beta + \beta^2 + \cdots) &= (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}, \\
c_t^\nu \sum_{j=0}^{\infty} \beta^j &= (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}, \\
\frac{c_t^\nu}{1-\beta} &= (1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}, \\
c_t^\nu &= (1-\beta)\left[(1+r)b_{t-1}^\nu + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}}\right].
\end{aligned}$$

Step 2 Decide upon the aggregation procedure. Having arrived at the consumption policy function for a given cohort, we now consider the aggregate consumption function (accounting for all cohorts). As the population, $N_t = (1+n)^t$, is clearly non-stationary, our aim is to calculate the aggregate level of household consumption, *per person*.

Step 3 Apply the procedure to consumption (LHS of cohort policy function). We already know the size of each cohort, and total population:

$$\text{Aggregate consumption, per person} = \frac{\sum_{\text{Cohort } \nu=0}^{\nu=t} \text{Size of cohort } \nu \times \text{Consumption of cohort } \nu}{\text{Total Population}}.$$

Thus we have that:

$$c_t \equiv \frac{c_t^0 + nc_t^1 + n(1+n)c_t^2 + \cdots + n(1+n)^{t-1}c_t^t}{(1+n)^t},$$

which is a simple linear aggregation of the consumption policy functions for each cohort, given above.

Step 4 Perform this aggregation. To compute this, note:

$$c_t = (1-\beta)(1+r)\frac{\left[b^0_{t-1} + nb^1_{t-1} + n(1+n)b^2_{t-1} + \cdots + n(1+n)^{t-1}b^t_{t-1}\right]}{(1+n)^t}$$
$$+ (1-\beta)\frac{1+n+n(1+n)+\cdots+n(1+n)^{t-1}}{(1+n)^t}\sum_{s=t}^{\infty}\frac{y_s-\tau_s}{(1+r)^{s-t}},$$

$$c_t = (1-\beta)(1+r)\frac{\left[b^0_{t-1} + nb^1_{t-1} + n(1+n)b^2_{t-1} + \cdots + n(1+n)^{t-1}b^t_{t-1}\right]}{(1+n)^t} + (1-\beta)\sum_{s=t}^{\infty}\frac{y_s-\tau_s}{(1+r)^{s-t}},$$

$$c_t = (1-\beta)(1+r)b_{t-1} + (1-\beta)\sum_{s=t}^{\infty}\frac{y_s-\tau_s}{(1+r)^{s-t}},$$

$$c_t = (1-\beta)\left[(1+r)b_{t-1} + \sum_{s=t}^{\infty}\frac{y_s-\tau_s}{(1+r)^{s-t}}\right],$$

where we have defined aggregate financial assets (in the third step) in a similar fashion to consumption:

$$b_{t-1} = \frac{b^0_{t-1} + nb^1_{t-1} + n(1+n)b^2_{t-1} + \cdots + n(1+n)^{t-2}b^{t-1}_{t-1}}{(1+n)^t}.$$

Imposing these two definitions, we refer to the final result as the aggregate consumption policy function.

Step 5 Apply **the same** aggregation to the period-t budget constraints.

$$b^\nu_t + c^\nu_t = (1+r)b^\nu_{t-1} + y_t - \tau_t,$$

$$c^\nu_t = (1+r)b^\nu_{t-1} + y_t - \tau_t - b^\nu_t,$$

$$c_t \equiv \frac{c^0_t + nc^1_t + n(1+n)c^2_t + \cdots + n(1+n)^{t-1}c^t_t}{(1+n)^t} = (1+r)\frac{b^0_{t-1} + nb^1_{t-1} + n(1+n)b^2_{t-1} + \cdots + n(1+n)^{t-1}b^t_{t-1}}{(1+n)^t}$$
$$+ \frac{1+n+n(1+n)+\cdots+n(1+n)^{t-1}}{(1+n)^t}[y_t - \tau_t]$$
$$- \frac{b^0_t + nb^1_t + n(1+n)b^2_t + \cdots + n(1+n)^{t-1}b^t_t}{(1+n)^t},$$

where we replace the term in red, b_{t-1}, with the same definition as above. Simplifying:

$$c_t = (1+r)b_{t-1} + y_t - \tau_t - \frac{b_t^0 + nb_t^1 + n(1+n)b_t^2 + \cdots + n(1+n)^{t-1}b_t^t + 0}{(1+n)^t}\frac{1+n}{1+n},$$

$$c_t = (1+r)b_{t-1} + y_t - \tau_t - \frac{b_t^0 + nb_t^1 + n(1+n)b_t^2 + \cdots + n(1+n)^{t-1}b_t^t + n(1+n)^t b_t^{t+1}}{(1+n)^t}\frac{1+n}{1+n},$$

$$c_t = (1+r)b_{t-1} + y_t - \tau_t - (1+n)\frac{b_t^0 + nb_t^1 + n(1+n)b_t^2 + \cdots + n(1+n)^{t-1}b_t^t + n(1+n)^t b_t^{t+1}}{(1+n)^{t+1}},$$

$$c_t = (1+r)b_{t-1} + y_t - \tau_t - (1+n)b_t,$$

where we note that to shift the definition of b_{t-1} forwards by one period, to b_t, we must add an additional term (though this is identically equal to zero due to our assumptions on bequests), and divide by an additional $(n+1)$. This is shown in blue.

Step 5 Conclude. After the linear aggregation procedure we are left with two conditions. One describes the aggregate consumption policy function:

$$c_t = (1-\beta)\Big[(1+r)b_{t-1} + \sum_{s=t}^{\infty}\frac{y_s - \tau_s}{(1+r)^{s-t}}\Big].$$

The other describes the evolution of aggregate financial assets, conditional on aggregate consumption.

$$(1+n)b_t + c_t = (1+r)b_{t-1} + y_t - \tau_t.$$

Clearly, we can then combine to eliminate consumption, to describe the evolution of financial assets in terms of exogenous variables.

2 Problem Set 2: Perpetual Youth (Blanchard Model)

Assume we have identified the following four properties of the model:

1. Total population, N_t and, within this, the size of each cohort, ν.

$$N_t = \underset{\nu=t}{1} + \underset{\nu=t-1}{\gamma} + \underset{\nu=t-2}{\gamma^2} + \cdots = \frac{1}{1-\gamma}.$$

2. The period-t budget constraint facing cohort ν:

$$b_t^\nu + c_t^\nu = \frac{1+r}{\gamma} b_{t-1}^\nu + y_t^\nu - \tau_t^\nu.$$

3. The intertemporal budget constraint (IBC) for a given cohort, ν:

$$\sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} c_s^\nu = \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu).$$

4. The Euler equation for a given cohort, ν:

$$c_{t+1}^\nu = \beta(1+r) c_t^\nu.$$

Step 1 Combine the Euler equation and intertemporal budget constraint to produce the consumption policy function for a given cohort, ν:

$$\begin{aligned}
\sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} c_s^\nu &= \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu), \\
c_t^\nu + \frac{\gamma}{1+r} c_{t+1}^\nu + \frac{\gamma^2}{(1+r)^2} c_{t+2}^\nu + \cdots &= \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu), \\
c_t^\nu + \frac{\beta(1+r)\gamma}{1+r} c_t^\nu + \frac{\beta^2(1+r)^2\gamma^2}{(1+r)^2} c_t^\nu + \cdots &= \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu), \\
c_t^\nu + \beta\gamma c_t^\nu + \beta^2\gamma^2 c_t^\nu + \cdots &= \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu), \\
c_t^\nu (1 + \beta\gamma + (\beta\gamma)^2 + \cdots) &= \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu), \\
c_t^\nu \sum_{j=0}^{\infty} (\beta\gamma)^j &= \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu), \\
\frac{c_t^\nu}{1-\beta\gamma} &= \frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu), \\
c_t^\nu &= (1-\beta\gamma)\Big[\frac{1+r}{\gamma} b_{t-1}^\nu + \sum_{s=t}^{\infty} \left(\frac{\gamma}{1+r}\right)^{s-t} (y_s^\nu - \tau_s^\nu)\Big].
\end{aligned}$$

Step 2 Decide upon the aggregation procedure. Having arrived at the consumption policy function for a given cohort, we now consider the aggregate consumption function (accounting for all cohorts). As the population, $N_t = \frac{1}{1-\gamma}$, is *stationary*, our aim is to calculate simply the aggregate level of household consumption.

Step 3 Apply the procedure to consumption (LHS of cohort policy function). We already know the size of each cohort:

$$\text{Aggregate consumption} = \sum_{\text{Cohort } \nu=0}^{\nu=t} \text{Size of cohort } \nu \times \text{Consumption of cohort } \nu.$$

Thus we have that:

$$c_t \equiv c_t^t + \gamma c_t^{t-1} + \gamma^2 c_t^{t-2} + \cdots ,$$

which is a simple linear aggregation of the consumption policy functions for each cohort, given above.

Step 4 Perform this aggregation:

$$\begin{aligned} c_t = (1-\beta\gamma)\frac{1+r}{\gamma}\Big[b_{t-1}^t + \gamma b_{t-1}^{t-1} + \gamma^2 b_{t-1}^{t-2} + \cdots\Big] \\ + (1-\beta\gamma)\sum_{s=t}^{\infty}\left(\frac{\gamma}{1+r}\right)^{s-t}\Big[y_s^t + \gamma y_s^{t-1} + \gamma^2 y_s^{t-2} + \cdots\Big] \\ - (1-\beta\gamma)\sum_{s=t}^{\infty}\left(\frac{\gamma}{1+r}\right)^{s-t}\Big[\tau_s^t + \gamma \tau_s^{t-1} + \gamma^2 \tau_s^{t-2} + \cdots\Big] \end{aligned}$$

$$c_t = (1-\beta\gamma)\Big[(1+r)b_{t-1} + \sum_{s=t}^{\infty}\frac{y_s - \tau_s}{(1+r)^{s-t}}\Big],$$

where we have defined:

$$\begin{aligned} \gamma b_{t-1} &\equiv b_{t-1}^t + \gamma b_{t-1}^{t-1} + \gamma^2 b_{t-1}^{t-2} + \cdots, \\ y_t &\equiv y_s^t + \gamma y_s^{t-1} + \gamma^2 y_s^{t-2} + \cdots, \\ \tau_t &\equiv \tau_s^t + \gamma \tau_s^{t-1} + \gamma^2 \tau_s^{t-2} + \cdots. \end{aligned}$$

Again, after imposing these definitions, we refer to the final result as the aggregate consumption policy function.

Step 5 Apply **the same** aggregation to the period-t budget constraints.

$$b_t^\nu + c_t^\nu = \frac{1+r}{\gamma} b_{t-1}^\nu + y_t^\nu - \tau_t^\nu,$$

$$c_t^\nu = \frac{1+r}{\gamma} b_{t-1}^\nu + y_t^\nu - \tau_t^\nu - b_t^\nu,$$

$$\begin{aligned} c_t \equiv c_t^t + \gamma c_t^{t-1} + \gamma^2 c_t^{t-2} + \cdots = \frac{1+r}{\gamma} & \left[b_{t-1}^t + \gamma b_{t-1}^{t-1} + \gamma^2 b_{t-1}^{t-2} + \cdots \right] \\ & + \left[y_t^t + \gamma y_t^{t-1} + \gamma^2 y_t^{t-2} + \cdots \right] - \left[\tau_t^t + \gamma \tau_t^{t-1} + \gamma^2 \tau_t^{t-2} + \cdots \right] \\ & - \left[b_t^t + \gamma b_t^{t-1} + \gamma^2 b_t^{t-2} + \cdots \right], \end{aligned}$$

where we replace the known terms with the same definition as above. Simplifying:

$$c_t = (1+r) b_{t-1} + y_t - \tau_t - \left[b_t^t + \gamma b_t^{t-1} + \gamma^2 b_t^{t-2} + \cdots \right].$$

Next, let us explicitly consider the definition of b_{t-1}. Repeating the definition:

$$\gamma b_{t-1} \equiv b_{t-1}^t + \gamma b_{t-1}^{t-1} + \gamma^2 b_{t-1}^{t-2} + \cdots,$$

we are able to shift this forwards by one period:

$$\gamma b_t \equiv b_t^{t+1} + \gamma b_t^t + \gamma^2 b_{t+1}^{t-1} + \gamma^3 b_{t+2}^{t-2} + \cdots,$$

and then note that $b_t^{t+1} = 0$ from our assumption on bequests. Hence:

$$\begin{aligned} \gamma b_t &\equiv \gamma b_t^t + \gamma^2 b_{t+1}^{t-1} + \gamma^3 b_{t+2}^{t-2} + \cdots, \\ b_t &\equiv b_t^t + \gamma b_{t+1}^{t-1} + \gamma^2 b_{t+2}^{t-2} + \cdots. \end{aligned}$$

The aggregate period-t budget constraint then becomes:

$$c_t = (1+r) b_{t-1} + y_t - \tau_t - b_t.$$

Step 5 Conclude. After the linear aggregation procedure we are left with two conditions. One describes the aggregate consumption policy function:

$$c_t = (1 - \beta\gamma) \left[(1+r) b_{t-1} + \sum_{s=t}^{\infty} \frac{y_s - \tau_s}{(1+r)^{s-t}} \right],$$

The other describes the evolution of aggregate financial assets, conditional on aggregate consumption.

$$c_t = (1 + r)b_{t-1} + y_t - \tau_t - b_t.$$

Clearly, we can then combine to eliminate consumption, to describe the evolution of financial assets in terms of exogenous variables.