International Economics, Lecture 1
Intertemporal Trade and the Current Account

Dan Wales
University of Cambridge

London School of Economics
1 October, 2018
Course Overview (Michaelmas Term)

- Three broad topics in international macroeconomics.
  1. Real: Current account and international RBC.
  2. Nominal: Monetary policy and international pricing.
  3. Financial crises and the nominal exchange rate.

- Objectives:
  - Recent developments in international macroeconomics.
  - Develop tools and ideas for writing research papers.

- Deliverables:
  - Weekly problem sets (two marked at random during the term).
  - Exam (January).
  - Extended essay (should you choose so).
Cumulative Current Account Balances

Why Focus on the Current Account?

- Indicator of **sustainability**.
  - Conventional wisdom: Persistent current account deficits above **5%** of GDP should be alarming.
    - Reversals are usually associated with **slower income growth**...
    - ...and large currency **depreciations**.

- Can this country generate future trade surpluses to repay debt burden? Ultimately: is the country solvent?

- Indicator of **macroeconomic imbalances**.
  - Clearly highlights which countries are **reliant** on external financing, and imbalance between countries.
  - But **causality** is unclear.

- Highlight **intertemporal trade**.
Current Account Balances

Source: IMF April 2018 WEO.
Plan

- Basic Definitions
- Two Period Endowment Model (SOE)
- Two Period Production Model (SOE)
- Two Period Model (Two Country)
- Dynamics of the Current Account
- Stochastic Infinite Horizon Model
- Engel and Rogers (2006)*
Basic Definitions

▶ Balance of Payments.
  ▶ Records transactions with the rest of the world.
  ▶ Double entry bookkeeping, as each transaction enters twice.
  ▶ Comprised of three separate accounts:

\[ \text{BoP} = CA + KA - FA = 0. \]

▶ Current Account (CA):
  ▶ Trade balance and net factor income from abroad.
  \[ ca_t \equiv nx_t + rb_{t-1}. \]

▶ Capital Account (KA):
  ▶ Net unilateral capital transfers from abroad (Small).

▶ Financial Account (-FA):
  ▶ Net acquisition of foreign financial assets.
  \[ fa_t \equiv b_t - b_{t-1}. \]
Three Current Account Relationships

- **Definitional:**
  \[ ca_t = nx_t + rb_{t-1} : \]
  as the trade balance and net factor income from abroad.

- **Change in net foreign assets:**
  \[ ca_t = fa_t = b_t - b_{t-1}, \]
  which uses the BoP relationship (intertemporal approach). Arises as unilateral payments in the capital account are small.

- **Difference between savings and investment:**
  \[ y_t = c_t + i_t + g_t + nx_t, \]
  \[ y_t = c_t + i_t + g_t + ca_t - rb_{t-1}, \]
  \[ ca_t = rb_{t-1} + y_t - c_t - \tau_{t+s}^{sp} + \tau_{t+s}^{sg} - g_t - i_t, \]
  which uses the GDP and BoP identities.
Two Period Endowment Model (SOE)

- Small Open Economy (SOE) takes world interest rate, $r$, as given.
- Representative household maximises lifetime utility:

$$U = u(c_1) + \beta u(c_2),$$

where $c_t$ represents real consumption, $\beta \in (0, 1)$ is the discount factor and the period utility function, $u(\cdot)$, obeys standard properties $[u'(\cdot) >, u''(\cdot) < 0]$.

- Subject to period-$t$ budget constraints:

$$b_t + c_t = y_t + (1 + r)b_{t-1},$$

for $t = \{1, 2\}$ where $b_0 = b_2 = 0$ and endowments, $y_t$, are perishable.
Intertemporal Budget Constraint

- Write down the period-\(t\) constraints:

\[
\begin{align*}
  b_1 + c_1 &= y_1 + (1 + r)b_0, \\
  b_2 + c_2 &= y_2 + (1 + r)b_1.
\end{align*}
\]

- Recall \(b_0 = b_2 = 0\):

\[
\begin{align*}
  b_1 + c_1 &= y_1, \\
  c_2 &= y_2 + (1 + r)b_1,
\end{align*}
\]

- Eliminate \(b_1\) to show that:

\[
c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r}.
\]

- The present discounted value of lifetime consumption equals the present discounted value of lifetime income.
Solution

- Substitute the IBC to rewrite the household problem as:

\[
\max_{c_1} u(c_1) + \beta u(y_2 + (1 + r)(y_1 - c_1)).
\]

- Differentiate to show the first order condition:

\[
\frac{u'(c_1)}{\beta u'(c_2)} = 1 + r.
\]

- The **marginal rate of substitution** between \(c_1\) and \(c_2\) is equal to the **relative price** of current consumption vis-a-vis future consumption.
Autarky vs Openness

- Under autarky household consumes income:
  \[ c_t = y_t, \]
  \[ b_t = 0. \]

- In a trading equilibrium, in general:
  \[ c_t \neq y_t, \]
  \[ b_t \neq 0. \]

Therefore define autarky real interest rate as:

\[ \frac{u'(y_1)}{\beta u'(y_2)} = 1 + r^{Aut}. \]
Suppose \( r^{Aut} > r \)

- Rearrange budget constraints:

\[
c_1 = y_1 - b_1, \\
c_2 = y_2 + (1 + r)b_1.
\]

- Then, using the FOC of the household problem:

\[
1 + r^{Aut} \equiv \frac{u'(y_1)}{\beta u'(y_2)} > \frac{u'(c_1)}{\beta u'(c_2)} = 1 + r.
\]

- Reveals:

\[
\frac{u'(y_1)}{\beta u'(y_2)} > \frac{u'(y_1 - b_1)}{\beta u'(y_2 + (1 + r)b_1)},
\]

such that \( r^{Aut} > r \) [\( r^{Aut} < r \)] implies an initial current account deficit [surplus], as \( b_1 < 0 \).
Graphical Representation - Autarky

- Initially consume endowments and receive some autarky level of utility, $u^{Aut}$. 

![Graphical representation of autarky with initial consumption points and utility level](image-url)
Then, introduce possibility to trade using intertemporal budget constraint.

\[ c_2 = y_2 - (1 + r)(c_1 - y_1) \]
Graphical Representation - New Equilibrium

- Under free trade, the small open economy moves to the new equilibrium point on a higher indifference curve, $u^{FT}$. 

![Graph showing indifference curves and an equilibrium point]
Graphical Representation - Current Accounts

- Same diagram also shows the current account in each period. Borrowing today, repaying tomorrow.
Initially the country was relatively well endowed with home endowment in the second period.

At the prevailing world interest rate, $r$, they therefore wish to consume a little more today, and a little less tomorrow.

Under free trade, they run a current account deficit today $CA_1 < 0$ such that $c_1 > y_1$.

... and a current account surplus tomorrow, $CA_2 > 0$, such that $c_2 < y_2$.

Together this smooths consumption between periods, relative to the autarky case.

Note: $nx_2 = -(1 + r)nx_1$, borrowing is repaid with interest.
Consider the model just described and assume \((1 + r)\beta = 1\):

\[
\frac{u'(c_1)}{\beta u'(c_2)} = 1 + r \rightarrow c_1 = c_2,
\]

and we therefore have \textbf{perfect consumption smoothing}.

Use the IBC to find consumption:

\[
c_1 = c_2 = \frac{(1 + r)y_1 + y_2}{2 + r}.
\]

Use the period budget constraint to solve for current account:

\[
b_1 = y_1 - c_1 = \frac{y_1 - y_2}{2 + r}.
\]

Also assume \(y_1 = y_2\) such that, initially, \(b_1 = 0\).
Permanent Shock

- Permanent increase in income: $\Delta y_1 = \Delta y_2 > 0$.

- Consumption in both periods **increases** to match the new level of discounted life-time income:

  $$
  \Delta c_1 = \Delta c_2 = \frac{(1 + r)\Delta y_1 + \Delta y_2}{2 + r} > 0.
  $$

- **No change** to the current account, which is still balanced:

  $$
  \Delta b_1 = \Delta y_1 - \Delta c_1 = \frac{\Delta y_1 - \Delta y_2}{2 + r} = 0.
  $$
Temporary Shock

- Now consider a temporary increase in income: \( \Delta y_1 > 0 \) but \( \Delta y_2 = 0 \).

- Again, consumption in both periods increases to match the new level of discounted life-time income:

\[
\Delta c_1 = \Delta c_2 = \frac{(1 + r)\Delta y_1}{2 + r} > 0,
\]

due to the change in \( y_1 \) only.

- However now the current account moves to **surplus**, as some of the temporary windfall is saved to **smooth consumption** over time:

\[
\Delta b_1 = \Delta y_1 - \Delta c_1 = \frac{\Delta y_1}{2 + r} > 0.
\]
“If you lose your lunch money one day, it’s not a problem. You simply borrow from a friend. Next time, you pay for his lunch. However if your parents cut your monthly allowance, you will have to change spending plans accordingly” [SGU, forthcoming].

It is hard to identify whether a change in income is temporary or permanent. One exception to this are natural disasters.
Two Period Production Model (SOE)

- Now introduce production and government spending to the previous model. Each period, government purchases satisfy:

\[ g_t = \tau_t, \]

where \( g_t \) is government spending and \( \tau_t \) represent lump-sum taxes, faced by households.

- Production technology follows:

\[ y_t = A_t F(k_t), \]

where \( F(\cdot) \) obeys standard properties. \([F(0) = 0, F'(\cdot) >, F''(\cdot) < 0]\).

- Capital accumulation:

\[ k_{t+1} = k_t + i_t, \]

where there is no depreciation, \( k_1 \) is given and investment may be negative.
New Intertemporal Budget Constraint

- The series of period-$t$ budget constraints now become:

$$b_t + c_t = y_t + (1 + r)b_{t-1} - i_t - \tau_t,$$

where $\tau_t$ represent lump-sum taxes and $i_t$ is investment.

- Again, eliminate $b_1$ and set $b_0 = b_2 = 0$ to show that:

$$c_1 + i_1 + \frac{c_2 + i_2}{1 + r} = y_1 - \tau_1 + \frac{y_2 - \tau_2}{1 + r}.$$

- PDV of lifetime expenditure on consumption and investment equals the PDV of lifetime income, after tax.

- To move to equilibrium notice:
  - $k_3 = 0 \rightarrow i_2 = -k_2$ and $i_1 = k_2 - k_1$, with $k_1$ given.

  - Taxation is specified exogenously as government consumption.

  - Production functions: $y_t = A_t F(k_t)$. 
Optimality Conditions

- Combine and rearrange for $c_2$:

$$c_2 = (1 + r) \left[ A_1 F(k_1) - g_1 - c_1 - k_2 + k_1 \right] + A_2 F(k_2) - g_2 + k_2,$$

- Use in optimisation problem:

$$\max_{c_1, k_2} u(c_1) + \beta u \left( (1 + r) (A_1 F(k_1) - g_1 - c_1 - k_2 + k_1) + A_2 F(k_2) - g_2 + k_2 \right),$$

- FOCs:

$$u'(c_1) = \beta (1 + r) u'(c_2), \quad \text{(wrt } c_1)$$

$$A_2 F'(k_2) = r. \quad \text{(wrt } k_2)$$

- Separation of savings and investment decisions ($k_2$ independent of $u$).

- No crowding out ($k_2$ independent of $g_t$).
Graphical Representation - Autarky

- Initially consume endowments and receive some autarky level of utility, $u^{Aut}$. IC tangent to PPF.
Graphical Representation - Introduce Trade

Then, introduce possibility to trade using intertemporal budget constraint. BC is tangent to PPF.

\[ c_2 = y_2 - i_2 - (1 + r)(c_1 + i_1 - y_1) \]
Under free trade, households/consumption move(s) to the new equilibrium point on a higher indifference curve, $u^{FT}$. 
Firms/production move(s) to a new equilibrium point. The level of capital, $k_2$, may be read directly from the graph.
Graphical Representation - What Happened?

- Two distinct procedures:
  - In the first investment decisions (capital) are made to maximise the possible size of the economic pie for the Home country, given international prices.
  
  - In the second, given this level of income, international prices and the household discount factor, households choose consumption to maximise utility.
Two Period Model (Two Country)

- Two countries: Home and Foreign (denoted by \(*\)).

- Return to endowment setting. Global equilibrium now requires:

\[ y_t + y_t^* = c_t + c_t^*. \]

- May be rewritten in terms of household savings \( s_t \equiv y_t - c_t \):

\[ s_t + s_t^* = 0. \]

- Or in terms of the current account (savings minus investment, but no investment here):

\[ CA_t + CA_t^* = 0. \]

- The main change: world interest rate is now endogenous, ensuring equilibrium in the global asset market.
The savings schedule for each country may be plotted. Assume different autarky equilibrium real interest rates.
To balance the global level of household savings, the interest rate must fall for Home and rise for Foreign.
Indeed, we may quantify the precise level of $r$ by considering the equilibrium condition $CA_t = -CA_t^*$. 

Graphical Representation - Free Trade II (Endowment)
Graphical Representation - What Happened?

- Under autarky for each country we define $r^{Aut}$ and $r^{*,Aut}$ such that:

\[ s_t(r_t^{Aut}) = 0, \]
\[ s_t^*(r_t^{*,Aut}) = 0. \]

- We assumed the parameterisation gave $r^{Aut} > r^{*,Aut}$.

- We must have that $r^{Aut} > r^{W} > r^{*,Aut}$, else:
  - $r^{W} > r^{Aut} > r^{*,Aut} \rightarrow CA_t > 0$ and $CA_t^* > 0 \rightarrow CA_t + CA_t^* > 0$.
  - $r^{Aut} > r^{*,Aut} > r^{W} \rightarrow CA_t < 0$ and $CA_t^* < 0 \rightarrow CA_t + CA_t^* < 0$.

- Therefore we know that when $r^{Aut} > r^{W} > r^{*,Aut}$:
  - $CA_t < 0$, while $CA_t^* > 0$.
  - Precise level given by $CA_t + CA_t^* = 0$. 
Move to a Production Economy

- The analysis for the two country case may be extended for an economy with **production**.

- Notice, in this case, current accounts are given as the **balance between household savings and investment**, $CA_t = s_t - i_t$, such that the real interest rate will be determined by:

$$s_t + s^*_t = i_t + i^*_t,$$

$$CA_t + CA^*_t = 0.$$ 

- For both endowment and production economies a shock to one country will propagate internationally to the other through changes in the real interest rate.
Graphical Representation - Metzler Diagram (Production)

- Savings and investment schedules are plotted for each country. Assume different autarky equilibrium real interest rates.
To balance the global level of savings and investment, the real interest rate must fall for Home and rise for Foreign.
Again, we may quantify the precise level of $r$ by considering the equilibrium condition $CA_t = -CA^*_t$. 
The shock shifts the Foreign investment curve to the right, both countries affected. In one case both CA’s close.
Dynamics of the Current Account

- Move from a 2-period setting to infinite horizon. Can do this in two steps:
  - Set a finite horizon, $T$.
  - Investigate properties as $T \to \infty$.

- Initially extend to a $T$-period model:
  - Representative household maximises lifetime utility:
    \[
    U_t = u(c_t) + \beta u(c_{t+1}) + \ldots + \beta^T u(c_{t+T}) = \sum_{s=t}^{t+T} \beta^{s-t} u(c_s).
    \]

  - Subject to a series of period-$t$ budget constraints:
    \[
    b_t + c_t + i_t + g_t = y_t + (1 + r)b_{t-1},
    \]
    \[
    b_{t+1} + c_{t+1} + i_{t+1} + g_{t+1} = y_{t+1} + (1 + r)b_t,
    \]
    \[
    \ldots
    \]
    \[
    b_{t+T} + c_{t+T} + i_{t+T} + g_{t+T} = y_{t+T} + (1 + r)b_{t+T-1}.
    \]

  - Assume $r$ is constant over time.
As before rewrite the budget constraints in terms of NFA:

\[ b_t = y_t - [c_t + i_t + g_t] + (1 + r)b_{t-1}, \]
\[ b_{t+1} = \frac{y_{t+1} - [c_{t+1} + i_{t+1} + g_{t+1}]}{1 + r} + b_t, \]
\[ \quad \ldots \]
\[ b_{t+T} = \frac{y_{t+T} - [c_{t+T} + i_{t+T} + g_{t+T}]}{1 + r} + b_{t+T-1}. \]

Iteratively substitute out to obtain:

\[ \frac{b_{t+T}}{(1 + r)^T} = (1 + r)b_{t-1} + \sum_{s=t}^{t+T} \frac{y_s - [c_s + i_s + g_s]}{(1 + r)^s-t}. \]
Finite Horizon Intertemporal Budget Constraint II

- Our usual assumption of $b_t = 0$ and argument that $b_{t+T} = 0$ will still apply.

- May therefore rewrite the intertemporal budget constraint as:

$$\sum_{s=t}^{t+T} \frac{y_s}{(1 + r)^{s-t}} = \sum_{s=t}^{t+T} \frac{c_s + i_s + g_s}{(1 + r)^{s-t}}.$$  

Again, it is clear that the PDV of lifetime income will be used to cover the PDV of expenditure on consumption, investment and by the government.

- Could stop here and discuss how this extension to a longer time horizon affects the model. A useful discussion in OR.

Instead we proceed directly to the infinite horizon case.
Infinite Horizon Model

▶ Now allow $T \to \infty$.

▶ Representative household maximises lifetime utility:

$$ U(c_t) = \sum_{s=t}^{\infty} \beta^{s-t} u(c_s). $$

▶ Subject to the intertemporal budget constraint:

$$ \sum_{s=t}^{\infty} \frac{c_s + i_s + g_s}{(1 + r)^{s-t}} + \lim_{T \to \infty} \frac{b_{t+T}}{(1 + r)^T} = (1+r)b_{t-1} + \sum_{s=t}^{\infty} \frac{y_s}{(1 + r)^{s-t}}. $$
Transversality Condition

▶ We will assume the terminal **transversality condition** (TVC):

\[
\lim_{T \to \infty} \frac{b_{t+T}}{(1 + r)^T} = 0.
\]

▶ This is the result of two distinct notions:

1. Optimality condition. With \( u'(\cdot) > 0 \) it will never be optimal for households to allow assets to grow more quickly than the real interest rate, such that:

\[
\lim_{T \to \infty} \frac{b_{t+T}}{(1 + r)^T} \geq 0.
\]

2. No-Ponzi (Madoff) assumption:

\[
\lim_{T \to \infty} \frac{b_{t+T}}{(1 + r)^T} \leq 0.
\]

▶ Continue to normalise **initial** condition: \((1 + r)b_{t-1} = 0\).
Optimality Conditions

▶ Take FOCs of the infinite horizon problem, subject to initial and terminal conditions:

\[ u'(c_t) = \beta(1 + r)u'(c_{t+1}), \quad \text{(wrt } c_t, \text{ Euler Equation)} \]

\[ A_{t+1}F'(k_{t+1}) = r. \quad \text{(wrt } k_{t+1}) \]

▶ Will hold in every time period.
Analytical Solution for Consumption Profile?

- In general, no. Three special cases:
  1. \( \beta = \frac{1}{1+r} \), such that households fully consumption smooth.
  2. Iso-elastic utility function (log as special case).
  3. Stochastic income, with \( \beta = \frac{1}{1+r} \) and quadratic utility.
Consider a constant, $\tilde{x}_t$:

$$\sum_{s=t}^{\infty} \frac{\tilde{x}_t}{(1 + r)^{s-t}} = \sum_{s=t}^{\infty} \frac{x_s}{(1 + r)^{s-t}},$$

such that the NPV of this constant, $\tilde{x}_t$, is equal to the NPV of the steam over the variable $x_t$. Hence:

$$\frac{\tilde{x}_t}{1 - \frac{1}{1+r}} = \sum_{s=t}^{\infty} \frac{x_s}{(1 + r)^{s-t}},$$

$$\tilde{x}_t = \frac{r}{1 + r} \sum_{s=t}^{\infty} \frac{x_s}{(1 + r)^{s-t}},$$

and $\tilde{x}_t$ is said to be the annuity value of the series $x_t$. 

Special Case 1, $\beta = \frac{1}{1+r}$

- Assume $\beta = \frac{1}{1+r}$ and rewrite the Euler equation as:

$$u'(c_t) = \beta(1 + r)u'(c_{t+1}), \rightarrow u'(c_t) = u'(c_{t+1}),$$

such that we have full consumption smoothing, $c_t = c_{t+1}$.

- Define household wealth, $W_t$, as:

$$W_t = (1 + r)b_{t-1} + \sum_{s=t}^{\infty} \frac{y_s - [i_s + g_s]}{(1 + r)^{s-t}},$$

$$= (1 + r)b_{t-1} + \frac{1 + r}{r} [\tilde{y}_t - \tilde{i}_t - \tilde{g}_t].$$

- But, using IBC and assumption consumption is shown to be a constant fraction of lifetime wealth, $W_t$:

$$W_t = \sum_{s=t}^{\infty} \frac{c_s}{(1 + r)^{s-t}} = \frac{1 + r}{r} c_t.$$
Special Case 1, $\beta = \frac{1}{1+r}$ ctd.

- Using these results we observe:

$$c_t = r \cdot b_{t-1} + \left[ \tilde{y}_t - \tilde{i}_t - \tilde{g}_t \right],$$

such that consumption only responds to changes in the value of permanent income.

- Turing to the current account:

$$ca_t = r \cdot b_{t-1} + y_t - c_t - i_t - g_t,$$

$$ca_t = (y_t - \tilde{y}_t) - (i_t - \tilde{i}_t) - (g_t - \tilde{g}_t).$$

- Changes in the current account arise due to temporary deviations in income, investment and government spending from their permanent levels. Call this the fundamental equation of the current account.
Special Case 2, iso-elastic utility

➢ Assume:

\[ u(c_t) = \frac{c^{1-1/\sigma}}{1 - 1/\sigma}, \]

with \( \sigma > 0 \), and hence \( u'(c_t) = c_t^{-1/\sigma} \).

➢ Euler equation may be rewritten as:

\[ u'(c_t) = \beta (1 + r) u'(c_{t+1}), \rightarrow c_{t+1} = \beta^\sigma (1 + r)^\sigma c_t. \]

➢ Substitute to find value of lifetime wealth as:

\[ \mathcal{W}_t = \sum_{s=t}^{\infty} \frac{c_s}{(1 + r)^{s-t}} = \frac{c_t}{1 - \beta^\sigma (1 + r)^{\sigma-1}}. \]

\[ ^1 \text{Assume } \beta^\sigma (1 + r)^{\sigma-1} < 1. \]
Special Case 2, iso-elastic utility ctd.

- Thus:
  \[ c_t = \frac{r + \theta}{1 + r} W_t, \]
  where \( \theta \equiv 1 - \beta^\sigma (1 + r)^\sigma. \)

- And hence, with a slight change to before:
  \[ ca_t = (y_t - \tilde{y}_t) - (i_t - \tilde{i}_t) - (g_t - \tilde{g}_t) - \frac{\theta}{1 + r} W_t. \]

- Similar intuition to before, with the addition of a tilt factor, \( \theta \):
  - Consumption still a **constant fraction** of PDV lifetime wealth.
  - Current account arises through:
    1. Deviations of variables from their permanent level.
    2. Difference in household preferences, compared to the market real interest rate (tilt factor). More impatient \( \rightarrow \) consume more today \( \rightarrow \) CA deficit today.
Special Case 2, log utility

► Log utility is a special case of the above, with \( \sigma = 1 \).

► In this case the Euler condition will become:

\[
c_{t+1} = \beta (1 + r) c_t.
\]

► We may characterise consumption growth patterns using a relationship between \( \beta \) and \( (1 + r) \):

► If \( \beta > (1 + r) \) then we have \( c_{t+1} > c_t \) as household are relatively **patient** (compared to international financial markets).

► If \( \beta < (1 + r) \) then we have \( c_{t+1} < c_t \) as household are relatively **impatient**.

► If \( \beta = (1 + r) \) then we have \( c_{t+1} = c_t \) and we recover the **perfect consumption smoothing** profile of special case 1.

► Intuition may also be given for a fixed \( \beta \) and considering changes in \( r \).
Stochastic Infinite Horizon Model

- Introduce concepts allowing income to be stochastic:
  - Assume that $y_t$, $i_t$ and $g_t$ follow stochastic processes with $y_t \sim \mathcal{N}(\mu_y, \sigma_y^2)$, $i_t \sim \mathcal{N}(\mu_i, \sigma_i^2)$ and $g_t \sim \mathcal{N}(\mu_g, \sigma_g^2)$.

- Households have rational expectations:
  - Knows model’s structure and shocks.
  - Forecast errors are uncorrelated, after conditioning on the available information set.

- Rewrite the household’s optimisation problem as:

$$\max_{\{c_s\}_{s=t}} \mathbb{E}_t \sum_{t=1}^{\infty} \beta^{t-1} u(c_t),$$

s.t.\: \mathbb{E}_t \sum_{s=t}^{\infty} \frac{c_s + i_s + g_s}{(1 + r)^{s-t}} = (1 + r)b_{t-1} + \mathbb{E}_t \sum_{s=t}^{\infty} \frac{y_s}{(1 + r)^{s-t}},$$

and $\lim_{T \to \infty} \mathbb{E}_t \frac{b_{t+T}}{(1 + r)^T} = 0$. 
Optimality Conditions in Stochastic Model

- Euler equation:
  \[ u'(c_t) = \beta (1 + r) \mathbb{E}_t[u'(c_{t+1})]. \]

- Now “holds in expectation” and takes the uncertainty of income fluctuations into account.
Aside: Steady State Indeterminacy*

- A steady state arises when $x_t = x$, $\forall t, x$. (Long run).

- Three equations specify the equilibrium. In steady state these may be written as:

\[
\begin{align*}
    u'(c) &= \beta(1 + r)u'(c), & (EE) \\
    r &= AF'(k), & (FOC, k) \\
    c + g &= rb + AF(k). & (IBC)
\end{align*}
\]

- EE only restricts the real interest rate, as $\beta(1 + r) = 1$.

- From this, the FOC for capital pins down $k$.

- However, the IBC is then unable to distinguish between the level of consumption and international financial assets. The level of financial assets, $b$, is indeterminate and the steady state is compatible with any level of foreign assets.

Assume utility is quadratic:

\[ u(c_t) = c_t - \frac{\alpha}{2} c_t^2, \]

with \( \alpha > 0 \) such that \( u'(c_t) = 1 - \alpha c_t \).

The Euler equation takes the form of a random walk:\(^2\):

\[ u'(c_t) = \beta (1 + r) \mathbb{E}_t [u'(c_{t+1})] \Rightarrow c_t = \mathbb{E}_t c_{t+1}, \]

From (abuse of) previous notation we have that:

\[ \mathcal{W}_t = \frac{1 + r}{r} c_t = (1 + r) b_{t-1} + \frac{1 + r}{r} \left[ \tilde{y}_t - \tilde{i}_t - \tilde{g}_t \right], \]

where now \( \tilde{x}_t \equiv \frac{r}{1+r} \sum_{s=t}^{\infty} \mathbb{E}_t \frac{x_s}{(1+r)^{s-t}}. \)

---

\(^2\)Classic reference is Hall (1978).
With:
\[ c_t = r \cdot b_{t-1} + \left[ \tilde{y}_t - \tilde{i}_t - \tilde{g}_t \right], \]
we again have a version of the fundamental equation of the current account:
\[ ca_t = (y_t - \tilde{y}_t) - (i_t - \tilde{i}_t) - (g_t - \tilde{g}_t). \]
where changes are now due to variables varying from their expected levels, and the variances of shocks are irrelevant for consumption decisions (certainty equivalence).
Example - Persistent Income I

- Suppose endowment model, with income following an AR(1) process (SGU 2.2):

\[ y_t = (1 - \rho) \mu_y + \rho y_{t-1} + \varepsilon_t, \]

with \( \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2_\varepsilon) \) and \( \rho \in (-1, 1) \), empirically \( \rho > 0 \).

- Can rewrite:

\[ y_{t+s} - \mu_y = \rho(y_{t+s-1} - \mu_y) + \varepsilon_{t+s}. \]

- Implies:

\[ \mathbb{E}_t[y_{t+s} - \mu_y] = \rho^s(y_t - \mu_y), \]

- Hence:

\[ \tilde{y}_t = \frac{r}{1 + r} \sum_{s=t}^{\infty} \frac{\mathbb{E}_t y_{t+s}}{(1 + r)^{s-t}} = \mu_y + \frac{r}{1 + r - \rho}(y_t - \mu_y). \]
Example - Persistent Income II

- Using $\tilde{y}_t$, given initial conditions $y_{t-1}$, $b_{t-1}$ and an exogenous shock, $\varepsilon_t$, we may immediately write down the path for consumption and the current account:

$$c_t = r \cdot b_{t-1} + \tilde{y}_t = r \cdot b_{t-1} + \mu_y + \frac{r\rho(y_{t-1} - \mu_y) + r\varepsilon_t}{1 + r - \rho},$$

$$ca_t = y_t - \tilde{y}_t = \frac{\rho(1 - \rho)(y_{t-1} - \mu_y) + (1 - \rho)\varepsilon_t}{1 + r - \rho}.$$

- With temporary shocks ($0 < \rho < 1$)
  - Unexpected positive endowment shocks ($\varepsilon_t > 0$) leads to $ca > 0$.
  - Want to smooth consumption gains over time.

- With permanent shocks ($\rho = 1$)
  - No effect on current account.
  - Income socks reflected one-to-one in consumption.
  - Permanent income hypothesis at work (Friedman).
Example - Persistent Income III

- Homework: Go to Matlab, generate the shock.

Impulse Response to Exogenous Income Shock

Notes: $\mu_y = 1$, $\rho_H = 0.6$, $\rho_L = 0.4$, $r = 5\%$. Calibrated for 1% income shock in highly persistent case. $y_t$ and $c_t$ shown as % deviation from SS.
Engel and Rogers (2006)*

Question: Is the (large) US current account (deficit) an outcome of **optimising** behaviour?

Their answer: Perhaps.

Model: Two country current account model, tweaked to account for private forecasters expectations of future income growth.
US Share of AE GDP*

Key observation: concurrent increase in US GDP as a share of total, and increase in CA deficit since 1980s.

Simple Two Country Model*

- Predictions:
  - Suppose \( \mathbb{E}_t [\Delta y_{t+k}] > 0 \) for some \( k > 0 \).
  - Then, optimising behaviour suggests home country should borrow now and repay later (when they have higher income). Hence CA deficit \( ca_{t+k-s} < 0 \) (for \( 0 < s < k \)).
  - A low real interest rate would amplify this mechanism.

- Specifics:
  - Endowment economies, same log preferences, no government.
  - Equilibrium conditions (+ foreign):
    \[
    c_{t+1} = \beta (1 + r) c_t, \tag{EE}
    \]
    \[
    b_{t-1} = \mathbb{E}_t \left[ \sum_{s=t}^{\infty} \frac{c_s - y_s}{(1 + r)^{s-t}} \right], \tag{IBC}
    \]
    \[
    y_t^w = y_t + y_t^* = c_t + c_t^*. \tag{Resources}
    \]
Use Fractions of World GDP*

- Combine and rewrite the above 3 equations as:

\[ c_t = (1 - \beta) \left( (1 + r) b_{t-1} + y_t^w \mathbb{E}_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} \gamma_s \right] \right), \]

\[ c_t = (1 + r)(1 - \beta) b_{t-1} + y_t \Gamma_t / \gamma_t, \]

where \( \gamma_s = \frac{Y_t}{Y_w} \) is the home country share of world GDP and \( \Gamma_t = (1 - \beta) \mathbb{E}_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} \gamma_s \right]. \)

- The current account as a percentage of GDP becomes:

\[ \frac{ca_t}{y_t} = \frac{b_t - b_{t-1}}{y_t} = \frac{y_t - c_t + rb_{t-1}}{y_t}, \]

\[ \frac{ca_t}{y_t} = 1 - \frac{\Gamma_t}{\gamma_t} - [1 - \beta(1 + r)] \frac{b_{t-1}}{y_t}. \]
Economic Mechanism*

▶ Two key equations:

\[
\frac{ca_t}{y_t} = 1 - \Gamma_t/\gamma_t - [1 - \beta(1 + r)] \frac{b_{t-1}}{y_t},
\]

\[
\frac{c_t}{y_t} = (1 + r)(1 - \beta) \frac{b_{t-1}}{y_t} + \Gamma_t/\gamma_t.
\]

▶ Assume, initially, \( b_{t-1} = 0 \) and that \( \gamma_{t+k} \uparrow \) for some \( k >> 0 \). Then \( c_t \uparrow \) and \( ca_t \downarrow \).

▶ Note: The home country share of world output cannot grow indefinitely! The current account will rebalance at some point in the future.
Model Predictions*

- The model does well, after several “heroic assumptions” (see section 2.2) and using private forecasters expectations of US output growth to help dynamics.

Summary

- Introduced basic open economy concepts, and focus on current account.
- Develop both a two-period and infinite horizon model to describe current account dynamics.
- Three special cases to achieve an analytical solution.
- Application: US current account deficit in early 2000s (discussed findings in Engel and Rogers (2006)).

